CSCanada
Studies in Mathematical Sciences
Vol. 7, No. 1, 2013, pp. [15–21]
DOI: 10.3968/j.sms.1923845220130701.731

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
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The Tensor Products of Unital Quantales

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Supported by the National Natural Science Foundation of China (Grant No. 10871121) and the Engagement Award (2010041) and Dr. Foundation (2010QD-J024) of Xi'an University of Science and Technology, China.

Received: June 4, 2013/ Accepted: August 6, 2013/ Published: August 31, 2013

Abstract: In this paper, the definition of bimorphism of unital quantales is given, Base on the tensor product of completed lattice, the concrete forms of the tensor product of unital quantales is obatined, and some properties of their are discussed.

Key words: Unital quantales; Unital quantales bimorphism; Tensor product; Quantale congruence

LIANG, S. (2013). The Tensor Products of Unital Quantales. *Studies in Mathematical Sciences, 7*(1), 15–21. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220130701.731 DOI: 10.3968/j.sms.1923845220130701.731

1. INTRODUCTION

Quantale was introduced by C.J.Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras [1]. The systematic introduction of quantale theory came from the book [2], which written by K.I.Rosenthal in 1990. Quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring [4], linear logic [5] and so on. So, the quantale theory has aroused great interests of many researches [6–16].

Since tensor product is very important concept in many algebraic structures, and their tensor product have been studied systemically in [17–19]. In this paper, we shall study the some properties of tensor product of unital quantales. For notions and concepts concerned, but explained, please refer to [2,20].

2. PRELIMINARIES

Definition 2.1 [2] A *quantale* is a complete lattice Q with an associative binary operation "&" satisfying:

$$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i) \quad \text{and} \quad (\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a),$$

for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q, respectively.

A quantale Q is said to be *unital* if there is an element $u \in Q$ such that $u\&a = a\&u = a$ for all $a \in Q$.

Definition 2.2 [2] Let Q be a quantale and $a \in Q$.

(1) a is $right-sided$ if and only if $a\&1 \leq a$.

(2) a is $left-sided$ if and only if $1\&a \leq a$.

(3) a is $two-sided$ if and only if a is both right and left side.

(4) a is *idempotent* if and only if $a\&a = a$.

Definition 2.3 [2] Let Q and P be quantales. A function $f : Q \longrightarrow P$ is a homomorphism of quantale if f preserves arbitrary sups and the operation "&". If Q and P are unital, then f is unital homomorphism if in addition to being a homomorphism, it satisfies $f(u_Q) = u_P$, where u_Q and u_P are units of Q and P, respectively.

Definition 2.4 [2] Let Q be a quantales. A subset $S \subseteq Q$ is a *subquantale* of Q iff the inclusion $S \hookrightarrow Q$ is a quantale homomorphism, i.e., S is closed under sups and "&".

Take a set X which is nonempty, N is the set of natural number. Let $X^* = \{x_1x_2\cdot\cdots\cdot x_n \mid x_i \in X,\ n \in N^+\}$. Define the operation " $*$ " on X such that

$$\begin{aligned} & x_1x_2\cdot\cdots\cdot x_n * y_1y_2\cdot\cdots\cdot y_m \\ =& x_1x_2\cdot\cdots\cdot x_n y_1y_2\cdot\cdots\cdot y_m \\ & \text{forall}\ \ x_1x_2\cdot\cdots\cdot x_n,\ y_1y_2\cdot\cdots\cdot y_m \in X^*. \end{aligned}$$

Let $P(X^*)$ denote the set of all subset of X^*. Then $P(X^*)$is clearly a complete lattice. Define the operation "&" on $P(X^*)$ such that $A\&B = \{a*b \mid a \in A,\ b \in B\}$ for all $A, B \in P(X^*)$,

Theorem 2.5 [13] Let X be a nonempty set. Then $(P(X^*), \&, \emptyset)$ be free unital quantales generated by X.

Definition 2.6 Let Q_1, Q_2 and Q be quantales, a map $f : Q_1 \times Q_2 \longrightarrow Q$ is said to be a *bimorphism* if it satisfying:

$$\forall \{x_i\}_{i\in I} \subseteq Q,\ \forall \{y_j\}_{j\in J} \subseteq Q_2,\ \forall x, s \in Q_1,\ \forall y, t \in Q_2,$$

(i) $f(\bigvee\limits_{i\in I} x_i, y) = \bigvee\limits_{i\in I} f(x_i, y)$;

(ii) $f(x, \bigvee\limits_{j\in J} y_j) = \bigvee\limits_{j\in J} f(x, y_j)$;

(iii) $f(x\&s,\ t) = f(x,\ t)\&f(s,\ t)$, $f(x,\ y\&t) = f(x,\ y)\&f(x,\ t)$.

3. THE TENSOR PRODUCTS OF UNITAL QUANTALES

The present section is dedicated to the tensor product of unital quantales. we will show its existence, and some properties are discussed.

Definition 3.1 Let Q_1, Q_2 and $Q_1 \otimes Q_2$ be unital quantales, the map $\tau : Q_1 \times Q_2 \longrightarrow T$ be a bimorphism. We call $Q_1 \otimes Q_2$ be the *tensor product* of Q_1 and Q_2, such that for any unital quantales Q and bimorphism $f : Q_1 \times Q_2 \longrightarrow Q$, there exists a unique unital homomorphism $h_f : T \longrightarrow Q$ with $h_f \circ \tau = f$.i.e.the triangle commutes.

$$\begin{tikzcd} Q_1 \times Q_2 \arrow[r, "\tau"] \arrow[dr, "f"'] & T \arrow[d, "h_f"] \\ & Q \end{tikzcd}$$

Obviously, by definition 3.1, if the tensor product of two unital quantales exists, then it is unique up to isomorphism.

Theorem 3.2 Let Q_1, Q_2 be unital quantales, then tensor product $Q_1 \otimes Q_2$ exists, and it is up to isomorphisms, the quotient of the free unital quantales $P((Q_1 \times Q_2)^*)$ with respect to the unital quantales congruce generated by the set

$$\begin{aligned} R = &\{(\{(x\&y_1, z)\}, \{(x, z) * (y, z)\}), (\{(x, y_2\&z)\}, \{(x, y_2) * (x, z)\}), \\ &\left(\{\bigvee(A, y)\}, \bigcup_{a\in A}\{(a, y)\}\right), \left(\{(x, \bigvee B)\}, \bigcup_{b\in B}\{(x, b)\}\right) \\ &\mid A \subseteq Q_1, B \subseteq Q_2,\ x, y_1 \in Q_1,\ y_2, z \in Q_2\}. \end{aligned}$$

Proof. Let $\mu : Q_1 \times Q_2 \longrightarrow P((Q_1 \times Q_2)^*)$ be the inclusion mapping, $f : Q_1 \times Q_2 \longrightarrow Q$be a unital quantales bimorphism. Since $P((Q_1 \times Q_2)^*)$ be the free unital quantales generated by $Q_1 \times Q_2$, then there exists a unique unital quantales homomorphism $h_f : P((Q_1 \times Q_2)^*) \longrightarrow Q$ such that $h_f \circ \mu = f$ i.e. the triangle commutes

$$\begin{tikzcd} Q_1 \times Q_2 \arrow[r, "\mu"] \arrow[dr, "f"'] & P((Q_1 \times Q_2)^*) \arrow[d, "h_f"] \\ & Q \end{tikzcd}$$

Since f is aunital quantales bimorphism, we have that for all

$$x, y_1 \in Q_1,\ y \in Q_2,\ A \subseteq Q_1,\ B \subseteq Q_2,\ y_2,\ z \in Q_2,$$

$$h_f(\{(\bigvee A, y)\}) = h_f \circ \mu((\bigvee A, y)) = f(\bigvee A, y) = \bigvee_{a\in A} f(A, y)$$

$$= \bigvee_{a\in A} h_f \circ \mu(a, y) = h_f(\bigcup_{a\in A} \mu(a, y)) = h_f(\bigcup_{a\in A}\{(a, y)\}),$$

i.e.
$$h_f(\{(\bigvee A, y)\}) = h_f(\bigcup_{a\in A}\{(a,y)\}),$$
and by symmetry
$$h_f(\{(x, \bigvee B)\}) = h_f(\bigcup_{b\in B}\{(x,b)\}).$$

$$\begin{aligned} h_f(\{(x\&y_1,z)\}) &= h_f\circ\mu(x\&y_1,z) = f(x\&y_1,z) = f(x,z)\&f(y_1,z)\\ =&(h_f\circ\mu(x,z))\&(h_f\circ\mu(y_1,z)) = h_f(\mu(x,z)*\mu(y_1,z)) = h_f(\{(x,z)*(y_1,z)\}),\end{aligned}$$
i.e. $h_f(\{(x\&y_1,z)\}) = h_f(\{(x,z)*(y_1,z)\})$, and analogously
$$h_f(\{(x,y_2\&z)\}) = h_f(\{(x,y_2)*(x,z)\}).$$

Therefore
$$R\subseteq R_{h_f} = \{(x,y)\in P((Q_1\times Q_2)^*)\times P((Q_1\times Q_2)^*) \mid h_f(x) = h_f(y)\}.$$

For convenience of expression, let $T = P((Q_1\times Q_2)^*)/ < R >$ denote the quotient quantale of the free unital quantales with respect to the unital quantales congruence generated by the set R, the mapping $\pi : P((Q_1\times Q_2)^*) \longrightarrow T$ denotes the canonical epimorphism. We define the map $h'_f : T \longrightarrow Q$ such that $h'_f([X]) = h_f(X)$ for all $[X]\in T$.

For all $[X]\in T, Y\in[X]$, since $(X,Y)\in\langle R\rangle\subseteq R_{h_f}$, then $h_f(X) = h_f(Y)$. Hence h'_f is well defined.

Next, we will prove that is a unital quantales homomorphism, i.e. it preserves arbitrary join and operation &.

For all $\{[X_i]\}_{i\in I}\subseteq T,\ [X],[Y]\in T$, we have

(i) $h'_f(\bigvee_{i\in I}[X_i]) = h'_f([\bigcup_{i\in I}X_i]) = h_f(\bigcup_{i\in I}X_i) = \bigvee_{i\in I}h_f(X_i) = \bigvee_{i\in I}h'_f([X_i]);$

(ii) $h'_f([X]\&[Y]) = h'_f([X\&Y]) = h_f(X\&Y) = h_f(X)\&h_f(Y) = h'_f([X])\&h'_f([Y]);$

$\forall\ (x,\ y)\in Q_1\times Q_2$, then
$$\begin{aligned} &h'_f\circ\pi\circ\mu(x,y) = h'_f\circ\pi(\{x,y)\}) = h'_f([\{x,y)\}])\\ =&h_f(\{x,y)\}) = h_f\circ\mu(x,y) = f(x,y),\end{aligned}$$
i.e. $h'_f\circ\pi\circ\mu = h_f\circ\mu = f$.

Let $\tau = \pi\circ\mu$, for all $\{x_i\}_{i\in I},\ \{y_j\}_{j\in J}\subseteq Q_1,\ x,s\in Q_2,\ y,t\in Q_2$, we can see that

$$\begin{aligned} &\text{(i)}\quad \tau(\bigvee_{i\in I}x_i,y) = (\pi\circ\mu)(\bigvee_{i\in I}x_i,\ y) = \pi(\{(\bigvee_{i\in I}x_i,\ y)\})\\ &=[(\bigvee_{i\in I}x_i,\ y)] = [\{\bigvee_{i\in I}x_i\}\times\{y\}],\quad\text{and}\\ &\bigvee_{i\in I}(\tau(x_i,\ y)) = \bigvee_{i\in I}(\pi\circ\mu)(x_i,\ y) = \bigvee_{i\in I}\pi(\{x_i,\ y\})\\ &= \bigvee_{i\in I}[(x_i,\ y)] = [\bigcup_{i\in I}(x_i,\ y)].\end{aligned}$$

Since

$$(\{\bigvee_{i\in I} x_i\} \times \{y\},\ \bigcup_{i\in I}(x_i,\ y)) \in R \subseteq < R >,$$

then

$$[\bigcup_{i\in I}(x_i,\ y)] = [\{\bigvee_{i\in I} x_i\} \times \{y\}].$$

Hence

$$\tau(\bigvee_{i\in I} x_i,\ y) = \bigvee_{i\in I}(\tau(x_i,\ y)).$$

Similarly, we have

$$\tau(x,\ \bigvee_{j\in J} y_j) = \bigvee_{j\in J}(\tau(x,\ y_j)).$$

(ii) $\tau(x\&s,\ t) = (\pi \circ \mu)(x\&s,\ t) = \pi(\{(x\&s,\ t)\})$
$=\pi(\{(x,t) * (s,t)\}) = \pi(\{(x,\ t)\}\&\{(s,t)\}) = \pi(\mu(x,t)\&\mu(s,t))$
$=(\pi \circ \mu(x,t))\&(\pi \circ \mu(s,t)) = \tau(x,\ t)\&\tau(s,\ t).$

Similarly, we have

$$\tau(x,\ y\&t) = \tau(x,\ y)\&\tau(x,\ t).$$

Therefore the map τ is a unital quantales homomorphism.

It is easy to verify that $h'_f \circ \tau = h'_f \circ \pi \circ \mu = h_f \circ \mu = f$, i.e. the diagram commute.

$$\begin{tikzcd}
Q_1 \times Q_2 \arrow[rr, "\mu"] \arrow[dr, "\tau"] \arrow[ddr, "f"'] & & P((Q_1 \times Q_2)^*) \arrow[dl, "\pi"'] \arrow[ddl, "h_f"] \\
& T \arrow[d, "h'_f"] & \\
& Q &
\end{tikzcd}$$

At last, we will prove that h_f is a unique quantale homomorphism.

Assume that h''_f is a unital quantale homomorphism such that $h''_f \circ \tau = f$.

For all $[X] \in T$, then $h''_f([X]) = h''_f \circ \pi(X) = h_f(X) = h'_f \circ \pi(X) = h'_f([X])$, i.e. $h''_f = h'_f$.

Therefore, the tensor product $Q_1 \otimes Q_2$ exists, and it is up to isomorphisms the quotient $P((Q_1 \times Q_2)^*)/< R >$. □

Definition 3.3 Let Q_1, Q be unital quantale, $x \in Q_1, y \in Q_2$. We denote by $x \otimes y$ the image of the pair (x, y) under mapping τ, i.e. the congruence class $[(x, y)] = x \otimes y$, and we call $x \otimes y$ is *tensor*.

Theorem 3.4 Let Q_1 and Q_2 be unital quantale, then $Q_1 \otimes Q_2 = \{\bigvee((x_1 \otimes y_1)\&(x_2 \otimes y_2)\& \cdots (x_n \otimes y_n)) \mid x_n \in Q_1, y_n \in Q_2, n \in N^+\}$.

Let Q_1, Q_2 and Q_3 be unique quantales, we denote by $Hom(Q_1 \otimes Q_2, Q_3)$ the set of all unique quantale homomorphisms between $Q_1 \otimes Q_2$ and Q_3. $Hom_c(Q_1, Hom(Q_2, Q_3))$ the set of all complete lattice homomorphisms between Q_1 and $Hom(Q_2, Q_3)$.

$$\begin{aligned}
&\text{Define } \ \eta : Hom(Q_1 \otimes Q_2, Q_3) \longrightarrow Hom(Q_1, Hom(Q_2, Q_3))\\
&h \longmapsto h_- : Q_1 \longrightarrow Hom(Q_2,\ Q_3)\\
&x \longmapsto h_x : Q_2 \longrightarrow Q_3\\
&\qquad\qquad y \longmapsto h(x \otimes y)
\end{aligned}$$

Theorem 3.5 Let Q_1, Q_2 and Q_3 be unique quantales, then map η be a complete lattice homomorphism.

Proof. Let $h : Q_1 \otimes Q_2 \longrightarrow Q_3$ be a unique quantale homomorphism. For all $x \in Q$, define mapping $h_x : Q_2 \longrightarrow Q_3$ such that $h_x(y) = h(x \otimes y)$ for all $y \in Q_2$. Next, we will prove that $h_x$is a unique quantale homomorphism.

(i) $\forall \ \{y_i\}_{i \in I} \subseteq Q_2,$ then

$$h_x(\bigvee_{i\in I} y_i) = h(x \otimes (\bigvee_{i\in I} y_i)) = h(\tau(x, \bigvee_{i\in I} y_i)) = h(\bigvee_{i\in I} \tau(x, y_i))$$
$$= h(\bigvee_{i\in I} y_i(x \otimes y_i)) = \bigvee_{i\in I} h(x \otimes y_i) = \bigvee_{i\in I} h_x(y_i);$$

(ii) $\forall \ y_1, y_2 \in Q_2,$ then

$$h_x(y_1\&y_2) = h(x \otimes (y_1\&y_2)) = h(\tau(x, y_1\&y_2)) = h(\tau(x,\ y_1) \ \& \ \tau(x,\ y_2)) = h(x \otimes y_1)\&h(x \otimes y_2) = h_x(y_1)\&h_x(y_2).$$

Therefore $h_x \in Hom(Q_1,\ Q_2)$ for all $x \in Q_1$.

Next, we will verify that $h_- : Q_1 \longrightarrow Hom(Q_1, Q_2)$ is a complete lattice homomorphism, i.e. h_- preserves arbitrary joins.

For all $\{x_i\}_{i\in I} \subseteq Q_1,\ y \in Q_2$, then

$$h_{\bigvee_{i\in I} x_i}(y) = h((\bigvee_{i\in I} x_i) \otimes y) = h(\tau(\bigvee_{i\in I} x_i,\ y))$$
$$= h(\bigvee_{i\in I} \tau(x_i,\ y)) = \bigvee_{i\in I} h(x_i \otimes\ y) = \bigvee_{i\in I} h_{x_i}(y).$$

For all $\{f^i\}_{i\in I} \subseteq Hom(Q_1 \otimes Q_2,\ Q_3)$, $x \in Q_1,\ \ y \in Q_2$, then

$$\eta(\bigvee_{i\in I} f^i)(x)(y) = (\bigvee_{i\in I} f^i)(x \otimes y) = \bigvee_{i\in I} (f^i(x \otimes y))$$
$$= \bigvee_{i\in I} f^i_x(y) = (\bigvee_{i\in I} f^i_x)(y) = \bigvee_{i\in I} \eta(f^i)(x)(y).$$

Therefore, the map η is well defined and is a complete lattice homomorphism. By theorem 3.4, we can see that the map η is a injective. $\square$

REFERENCES

[1] Mulvey, C. J. (1986). &. *Suppl. Rend. Circ. Mat. Palermo Ser., 12*, 99–104.

[2] Rosenthal, K. I. (1990). *Quantales and their applications*. London: Longman Scientific and Technical.

[3] Nawaz, M. (1985). *Quantales:quantale sets*. Ph.D. Thesis, University of Sussex.

[4] Brown, C., & Gurr, D. (1993). A representation theorem for quantales. *Journal of Pure and Applied Algebra, 85*, 27–42.

[5] Resende, P. (2001). Quantales, finite observations and strong bisimulation. *Theoretical Computer Science, 254*, 95–149.

[6] Coniglio, M. E., & Miraglia, F. (2001). Modules in the category of sheaves over quantales. *Annals of Pure and Applied Logic, 108*, 103–136.

[7] Resende, P. (2002). Tropological systems are points of quantales. *Journal of Pure and Applied Algebra, 173*, 87–120.

[8] LI, Yongming, ZHOU, meng, & LI, Zhihui (2002). Projectives and injectives in the category of quantales. *Journal of Pure and Applied Algebra, 176*(2), 249–258.

[9] Picado, J. (2004). The quantale of Galois connections. *Algebra Universalis, 52*, 527–540.

[10] Resende, P. (2007). Etale groupoids and their quantales. *Advances in Mathematics, 208*(1), 147–209.

[11] Russo, C. (2007). *Quantale modules, with applications to logic and image processing*. Ph.D.Thesis, Salerno: University of Salerno.

[12] Solovyov, S. (2008). On the category Q-Mod. *Algebra Universalis, 58*(1), 35–58.

[13] LIU, Z. B., & ZHAO, B. (2006). Algebraic properties of category of quantale. *Acta Mathematica Sinica, Chinese Series, 49*(6), 1253–1258.

[14] ZHOU, Y. H., & ZHAO, B. (2006). The free objects in the category of involutive quantales and its property of well-powered. *Chinese Journal of Engineering Mathematics, 23*(2), 216–224 (In Chinese).

[15] HAN, S. W., & ZHAO, B. (2009). The quantic conuclei on quantales. *Algebra Universalis, 61*(1), 97–114.

[16] ZHAO, B., & LIANG, S. H. (2009). The categorgy of double quantale modules. *Acta Mathematica Sinica, Chinese Series, 52*(4), 821–832.

[17] Shmuely, Z. (1979). The tensor product of distributive lattices. *Algebra Universalis, 9*(3), 281–296.

[18] Lakser, H. (1981). The semilattice tensor product of projective distributive lattices. *Algebra Universalis, 13*(1), 78–8.

[19] Grätzer, G., &Wehrung, F. (2000). Tensor products of semilattices with zero, revisited. *Journal of Pure and Applied Algebra, 147*(3), 273–301.

[20] Adámek, J. (1990). *Abstrat and concrete categories*. New York: Wiley.