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Lacunary Statistical Convergence of Sequences of Sets

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Abstract: Several notions of convergence for subsets of metric space appear in the literature. In this paper we define lacunary statistical convergence for sequences of sets and study in detail the relationship between other convergence concepts.

Key words: Statistical convergence; Lacunary statistical convergence; Almost convergence; Sequence of sets; Wijsman convergence

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1. INTRODUCTION AND BACKGROUND

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. We shall define lacunary statistical convergence for sequences of sets and establish some basic results regarding these notions.

Let us start with fundamental definitions from the literature. The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|,$$

where $|k \leq n : k \in K|$ denotes the number of elements of K not exceeding n.

Statistical convergence of sequences of points was introduced by Fast (Fast, 1951). Schoenberg (Schoenberg, 1959) established some basic properties of statistical convergence and also studied the concept as a summability method.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st-\lim x_k = L$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = L$, then $st - \lim x_k = L$. The converse does not hold in general.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

The concept of lacunary statistical convergence was defined by Fridy and Orhan (Fridy & Orhan, 1993). A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{r} \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0. \tag{1}$$

In this case we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$.

The sequence space N_θ, which is defined by

$$N_\theta = \left\{(x_k) : \lim_{r} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0\right\}.$$

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a\in A} \rho(x, A).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this cace we write $W - \lim A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

The concepts of Wijsman statistical convergence and Wijsman strong Cesaro summability were introduced by Nuray and Rhoades (Nuray & Rhoades, 2012): Let (X, ρ) a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman statistically convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Let (X, ρ) a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman strongly Cesaro summable to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades (Nuray & Rhoades, 2012) as follows: Let (X, ρ) a metric space. For any non-empty closed subsets A_k of X, the sequence $\{A_k\}$ is said to be bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

2. MAIN RESULTS

In this section, we will define Wisjman lacunary statistical convergence of sequences of sets and will give the relationship between Wijsman statistical convergence and Wisjman lacunary statistical convergence of sequences of sets.

Definition 1. *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary summable to A if for each $x \in X$,*

$$\lim_{r} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_k) = d(x, A).$$

In this case we write $A_k \to A(WN_\theta)$.

The set of Wijsman lacunary summable sequences will be denoted

$$WN_\theta := \left\{ \{A_k\} : \lim_{r} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_k) = d(x, A) \right\}.$$

Definition 2. *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistically convergent to A if for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{r} \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

In this case we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

The set of Wijsman lacunary statistically convergent sequences will be denoted

$$WS_\theta := \left\{ \{A_k\} : S_\theta - \lim\ _W\ A_k = A \right\}.$$

Example 1. *Let $X = \mathbb{R}$ and we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \{x \in \mathbb{R} : 2 \leq x \leq k_r - k_{r-1}\} & , \ \textit{if } k \geq 2 \textit{ and } k \textit{ is square integer,} \\ \{1\} & , \ \textit{otherwise.} \end{cases}$$

This sequence is not Wijsman lacunary summable. But since

$$\lim_{r\to\infty} \frac{1}{h_r} |\,\{k \in I_r : |d(x, A_k) - d(x, \{1\})| \geq \varepsilon\}\,| = \lim_{r\to\infty} \frac{\sqrt{k_r - k_{r-1}}}{h_r} = 0,$$

this sequence is Wijsman lacunary statistically convergent to the set $A = \{1\}$.

Example 2. *Let $X = \mathbb{R}^2$ and we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \dfrac{1}{k}\right\} & , \quad \begin{array}{l} \text{if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}] \text{ and} \\ k \text{ is a square integer,} \end{array} \\ \{(0,0)\} & , \quad \text{otherwise.} \end{cases}$$

This sequence is Wijsman lacunary statistical converget to the set $A = \{(0,0)\}$ since

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_k) - d(x, \{(0,0)\})| \geq \varepsilon\}| = 0.$$

But it is not Wijsman lacunary summable.

Definition 3. *Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary strongly summable to A for each $x \in X$,*

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN_\theta])$.

The set of Wijsman lacunary strongly summable sequences will be denoted

$$[WN_\theta] := \left\{ \{A_k\} : \lim_r \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| = 0 \right\}.$$

Example 3. *Let $X = \mathbb{R}$ and we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \{x \in \mathbb{R} : 2 \leq x \leq k_r - k_{r-1}\} & , \quad \text{if } k \geq 2 \text{ and } k \text{ is square integer,} \\ \{1\} & , \quad \text{otherwise.} \end{cases}$$

This sequence is Wijsman lacunary strongly summable to the set $A = \{1\}$ since

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, \{1\})| = \lim_{r\to\infty} \frac{1}{h_r} . \sqrt{k_r - k_{r-1}} = 0.$$

i.e., $\{A_k\} \in [WN_\theta]$.

Theorem 1. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and A, A_k be non-empty closed subsets of X;*

(i) *(a)* $A_k \to A([WN_\theta]) \Rightarrow A_k \to A(WS_\theta)$

(b) $[WN_\theta]$ *is a proper subset of* WS_θ;

(ii) $\{A_k\} \in L_\infty$ *and* $A_k \to A(WS_\theta) \Rightarrow A_k \to A([WN_\theta])$;

(iii) $WS_\theta \cap L_\infty = [WN_\theta] \cap L_\infty$,

where L_∞ denotes the set of bounded sequences of sets.

Proof. $(i)-(a)$. if $\varepsilon>0$ and $A_k\to A([WN_\theta])$ we can write

$$\begin{aligned}\textstyle\sum_{k\in I_r}|d(x,A_k)-d(x,A)| &\geq \sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}|d(x,A_k)-d(x,A)|\\ &\geq \varepsilon.\left|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\end{aligned}$$

which gives the result.

$(i)-(b)$. In order to show that the inclusion $[WN_\theta]\subset WS_\theta$ in (i) is proper, let θ be given and we define a sequence $\{A_k\}$ as follows:

$$A_k=\begin{cases}\{k\} & , \quad \textit{if} \quad k_{r-1}<k\leq k_{r-1}+\left[\sqrt{h_r}\right] \qquad r=1,2,\cdots\\ \{0\} & , \quad \textit{otherwise.}\end{cases}$$

Note that $\{A_k\}$ is not bounded. We have, for every $\varepsilon>0$ and for each $x\in X$,

$$\frac{1}{h_r}|\{k\in I_r:|d(x,A_k)-d(x,\{0\})|\geq\varepsilon\}|=\frac{\left[\sqrt{h_r}\right]}{h_r}\to 0 \qquad \text{as } r\to\infty$$

i.e., $A_k\to\{0\}(WS_\theta)$. But,

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,\{0\}|=\frac{1}{h_r}\frac{\left[\sqrt{h_r}\right].\left(\left[\sqrt{h_r}\right]+1\right)}{2}\to\frac{1}{2}\neq 0$$

hence $A_k\nrightarrow A([WN_\theta])$.

(ii) Suppose that $A_k\to A(WS_\theta)$ and $A_k\in L_\infty$, say $|d(x,A_k)-d(x,A)|\leq M$ for each $x\in X$ and all k. Given $\varepsilon>0$, we get

$$\begin{aligned}&\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\ =\;&\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}|d(x,A_k)-d(x,A)|\\ &+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)|\\ \leq\;&\frac{M}{h_r}|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|+\varepsilon\end{aligned}$$

hence we have the result.

(iii) This is follows from consequences (i) and (ii). □

Lemma 1. *For any lacunary sequence $\theta=\{k_r\}$, $st-\lim_W A_k=A$ implies $S_\theta-\lim_W A_k=A$ if and only if* $\liminf_r q_r>1$.

Proof. Suppose first that $\liminf_r q_r>1$; then there exists a $\lambda>0$ such that $q_r\geq 1+\lambda$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\lambda}{1+\lambda}.$$

If $st-\lim_W A_k=A$, then for every $\varepsilon>0$ and for sufficiently large r, we have

$$\begin{aligned}&\frac{1}{k_r}|\{k\leq k_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|\\ \geq\;&\frac{1}{k_r}|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|\\ \geq\;&\frac{\lambda}{1+\lambda}.\left(\frac{1}{h_r}|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|\right)\end{aligned}$$

this proves the sufficiently.

Conversely, suppose that $\liminf_r q_r = 1$. Since θ is lacunary, we can select a subsequence $\left\{k_{r_j}\right\}$ of θ satisfying

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j} \qquad \text{and} \qquad \frac{k_{r_j-1}}{k_{r_{j-1}}} > j, \qquad \text{where } r_j \geq r_{j-1} + 2.$$

Now we define a sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} (x,y) \in \mathbb{R}^2, \quad x^2 + (y-1)^2 = \frac{1}{k^4} & , \quad k \in I_{r_j}, \\ \{(0,0)\} & , \quad \textit{otherwise.} \end{cases}$$

Then,

$$\frac{1}{h_{r_j}} \sum_{k \in I_{r_j}} |d(x, A_k) - d(x, \{(0,0)\})| = T \qquad \text{for } j = 1, 2, \cdots, \ (T \in \mathbb{R}^+)$$

and

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, \{(0,0)\})| = 0 \qquad \text{for } r \neq r_j.$$

It follows that $\{A_k\} \notin [WN_\theta]$. However, $\{A_k\}$ is Wijsman strongly Cesaro summable, since if n is any sufficient large integer can find the unique j for which $k_{r_j-1} < n < k_{r_{j+1}-1}$ and write

$$\frac{1}{n} \sum_{i=1}^{n} |d(x, A_k) - d(x, \{(0,0)\})| \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

As $n \to \infty$ it follows that also $j \to \infty$. Hence $\{A_k\} \in |W\sigma_1|$. The above Theorem 1 (ii) implies that $\{A_k\} \notin WS_\theta$, but it follows from (Nuray & Rhoades, 2012, theorem 17) that $\{A_k\} \in WS$. Hence $WS \nsubseteq WS_\theta$. ☐

Lemma 2. *For any lacunary sequence θ, $S_\theta - \lim_W A_k = A$ implies $st - \lim_W A_k = A$ if and only if* $\limsup_r q_r < \infty$.

Proof. If $\limsup_r q_r < \infty$,then there is an $K > 0$ such that $q_r < K$ for all r. Suppose that $S_\theta - \lim_W A_k = A$, and let $U_r := |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon|$. By (1), given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that

$$\frac{U_r}{h_r} < \varepsilon \qquad \text{for all } r > r_0.$$

Now let $M := \max\{U_r : 1 \leq r \leq r_0\}$ and let t be any integer satisfying $k_{r-1} < t \leq$

k_r; then we can write

$$\begin{aligned}
& \frac{1}{t}|\{k \le t : |d(x,A_k) - d(x,A)| \ge \varepsilon\}| \\
\le\ & \frac{1}{k_{r-1}}|\{k \le k_r : |d(x,A_k) - d(x,A)| \ge \varepsilon\}| \\
=\ & \frac{1}{k_{r-1}}\{U_1 + U_2 + \cdots + U_{r_0} + U_{r_0+1} + \cdots + U_r\} \\
\le\ & \frac{M}{k_{r-1}}.r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{U_{r_0+1}}{h_{r_0+1}} + \cdots + h_r\frac{U_r}{h_r}\right\} \\
\le\ & \frac{r_0.M}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{U_r}{h_r}\right)\{h_{r_0+1} + \cdots + h_r\} \\
\le\ & \frac{r_0.M}{k_{r-1}} + \varepsilon.\tfrac{k_r - k_{r_0}}{k_{r-1}} \\
\le\ & \frac{r_0.M}{k_{r-1}} + \varepsilon.q_r \le \frac{r_0.M}{k_{r-1}} + \varepsilon.K
\end{aligned}$$

and the sufficiently follows immediately.

Conversely, suppose that $\limsup_r q_r = \infty$ and construct a sequence in $[WN_\theta]$ that is not Wijsman strongly Cesaro summable. First select a subsequence (k_{r_j}) of the lacunary sequence $\theta = \{k_r\}$ such that $q_{r_j} > j$, and then we define a bounded sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \{1\} & , \quad k_{r_j-1} < k < 2k_{r_j-1}, \\ \{0\} & , \quad \text{otherwise.} \end{cases}$$

Then

$$\tau_{r_j} = \frac{1}{h_{r_j}}\sum_{I_{r_j}} |d(x,A_k) - d(x,\{0\})| = \frac{k_{r_j-1}}{k_{r_j-1} - k_{r_{j-1}}} < \frac{1}{j-1}$$

and, if $r \ne r_j$, $\tau_r = 0$. Thus $\{A_k\} \in [WN_\theta]$. Observe next that any sequence in $|W\sigma_1|$ consisting of only $\{0\}$'s and $\{1\}$'s has an associated Wijsman strongly limit $\{L\}$ which is $\{0\}$ or $\{1\}$. For the sequence $\{A_k\}$ above, and $k = 1, 2, ..., k_{r_j}$,

$$\frac{1}{k_{r_j}}\sum_k |d(x,A_k) - d(x,\{1\})| \ge \frac{1}{k_{r_j}}(k_{r_j} - 2k_{r_j-1}) = 1 - \frac{2k_{r_j-1}}{k_r} > 1 - \frac{2}{j}$$

which converges to $\{1\}$, and, for $k = 1, 2, ...2k_{r_j-1}$,

$$\frac{1}{2k_{r_j-1}}\sum_k |d(x,A_k) - d(x,\{0\})| \ge \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2}$$

and it follows that $\{A_k\} \notin |W\sigma_1|$. □

Combining Lemma 1 and Lemma 2 we have

Theorem 2. *Let θ be a lacunary sequence; then $WS = WS_\theta$ if and only if*

$$1 < \liminf_r q_r \le \limsup_r q_r < \infty;$$

then $st - \lim_W A_k = A$ implies $S_\theta - \lim A_k = A$.

Proof. This follows from Lemma 1 and Lemma 2. □

Theorem 3. *If* $\{A_k\} \in WS \cap WS_\theta$, *then* $S_\theta - \lim_W A_k = st - \lim_W A_k$.

Proof. Suppose $st - \lim_W A_k = A$ and $S_\theta - \lim_W A_k = B$, and $A \neq B$. For $\frac{1}{2}|d(x,A) - d(x,B)| > \varepsilon$ and each $x \in X$ we get

$$\lim_n \frac{1}{n}|\{k \leq n : |d(x,A_k) - d(x,B)| \geq \varepsilon\}| = 1.$$

Consider the k_m th term of the statistical limit expression $\frac{1}{n}|\{k \leq n : |d(x,A_k) - d(x,B)| \geq \varepsilon\}|$:

$$\begin{aligned} & \frac{1}{k_i}|\{k \leq k_i : |d(x,A_k) - d(x,B)| \geq \varepsilon\}| \\ = & \frac{1}{k_i}\left|\{k \in \bigcup_{r=1}^{i} I_r : |d(x,A_k) - d(x,B)| \geq \varepsilon\}\right| \\ = & \frac{1}{k_i}\sum_{r=1}^{i} |\{k \in I_r : |d(x,A_k) - d(x,B)| \geq \varepsilon\}| \\ = & \frac{1}{\sum_{r=1}^{i} h_r}\sum_{r=1}^{i} |\{k \in I_r : |d(x,A_k) - d(x,B)| \geq \varepsilon\}|, \\ = & \frac{1}{\sum_{r=1}^{i} h_r}\sum_{r=1}^{i} h_r u_r \end{aligned} \tag{2}$$

where $u_r = \frac{1}{h_r}|\{k \in I_r : |d(x,A_k) - d(x,B)| \geq \varepsilon\}| \to 0$ because $A_k \to B(WS_\theta)$. Since θ is lacunary sequence, (2) is a regular weighted mean transform of u, and therefore it, too, tends to zero as $i \to \infty$. Also, since this is a subsequence of $\left\{\frac{1}{n}|\{k \leq n : |d(x,A_k) - d(x,B)| \geq \varepsilon\}|\right\}_{n=1}^{\infty}$, we infer that

$$\lim_n \frac{1}{n}|\{k \leq n : |d(x,A_k) - d(x,B)| \geq \varepsilon\}| \neq 1,$$

and this contradiction shows that we cannot have $A \neq B$. □

We now consider the inclusion of $WS_{\theta'}$ by WS_θ, where θ' is lacunary refinement of θ. Recall (Freedman, Sember, & Raphael, 1978) that the lacunary sequence $\theta' = \{k'_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{k'_r\}$.

Theorem 4. *If* θ', *is a lacunary refinement of* θ *and* $A_k \to A(WS_{\theta'})$, *then* $A_k \to A(WS_\theta)$.

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < ... < k'_{r,v(r)} = k_r, \qquad \text{where } I'_{r,i} = (k'_{r,i-1}, k'_{r,1}].$$

Note that for all r, $v(r) \geq 1$ because $\{k_r\} \subseteq \{k'_r\}$. Let $\{I^*_j\}_{j=1}^{\infty}$ be the sequence of abutting intervals $\{I'_{r,i}\}$ ordered by increasing right end points. Since $A_k \to A(WS_{\theta'})$, we get, for each $\varepsilon > 0$,

$$\lim_j \sum_{I^*_j \subset I_r} \frac{1}{h^*_r} |\{k \in I^*_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0. \tag{3}$$

As before we write, $h_r = k_r - k_{r-1}$, $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k_{r-1}$. For each $\varepsilon > 0$ we have

$$\begin{aligned} & \frac{1}{h_r} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ = \;& \frac{1}{h_r} \sum_{I^*_j \subseteq I_r} h^*_j \frac{1}{h^*_j} |\{k \in I^*_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ = \;& \frac{1}{h_r} \sum_{I^*_j \subseteq I_r} h^*_j (C_{\theta'} \chi_K)_j \end{aligned} \tag{4}$$

where χ_K is the characteristic function of the set $K := \{k \in I^*_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}$ and $C_{\theta'} \chi_K$ where,

$$C_{\theta'} \chi_K = \begin{cases} \dfrac{1}{h^*_j}, & \textit{if } k \in I^*_j \\ 0, & \textit{if } k \notin I^*_j \end{cases}$$

By (3), $C_{\theta'} \chi_K$ is a null sequence, and (4) is a regular weighted mean transform of $C_{\theta'} \chi_K$. Hence, the transform (4) also to goes zero as $r \to \infty$. □

Nuray and Rhoades (Nuray & Rhoades, 2012) introduced the notion of strongly almost convergence for sequence of sets as follows:

Definition 4. *For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $x \in X$,*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0$$

uniformly in i. In this case we write $A_k \to A([WAC])$.

The set of Wijsman strongly almost convergent sequences will be denoted

$$[WAC] := \left\{ \{A_k\} : \lim_n \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0 \right\}.$$

It is known that

$$[WAC] \subset L_\infty. \tag{5}$$

Example 4. *Let $X = \mathbb{R}$ and we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \{x \in \mathbb{R} : 2 \leq x \leq k\} & , \textit{ if } k \geq 2 \textit{ and } k \textit{ is square integer,} \\ \{1\} & , \textit{ otherwise.} \end{cases}$$

This sequence is Wijsman strongly almost convergent to the set $A = \{1\}$ *since*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, \{1\})| \leq \lim_{n\to\infty} \frac{1}{n}.\sqrt{n} = 0$$

uniformly i. i.e., $\{A_k\} \in [WAC]$.

Proposition 1. $[WAC] = \bigcap WN_\theta$

Theorem 5. *If* Φ *denotes the set of all lacunary sequences, then*

$$[WAC] = L_\infty \cap \left(\bigcap_{\theta\in\Phi} WS_\theta \right).$$

Proof. By proposition above, the relations (5) and Theorem 1 (*iii*),

$$L_\infty \supset [WAC] = \bigcap_{\theta\in\Phi} [WN_\theta] = L_\infty \cap \left(\bigcap_{\theta\in\Phi} [WN_\theta] \right) = \bigcap_{\theta\in\Phi} (L_\infty \cap [WN_\theta])$$

$$= \bigcap_{\theta\in\Phi} (L_\infty \cap WS_\theta) = L_\infty \cap \left(\bigcap_{\theta\in\Phi} WS_\theta \right).$$

□

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