Progress in Applied Mathematics
Vol. 1, No. 2, 2011, pp. 43-70
www.cscanada.org

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net

A Logical Calculus to Intuitively and Logically Denote Number Systems

Pith XIE[1,2,*]

Abstract: Simple continued fractions, base-b expansions, Dedekind cuts and Cauchy sequences are common notations for number systems. In this note, first, it is proven that both simple continued fractions and base-b expansions fail to denote real numbers and thus lack logic; second, it is shown that Dedekind cuts and Cauchy sequences fail to join in algebraical operations and thus lack intuition; third, we construct a logical calculus and deduce numbers to intuitively and logically denote number systems.

Key Words: Number system; Logical calculus; Series

1. INTRODUCTION

Number system is a set together with one or more operations. Any notation for number system has to denote both set and operations. The common notations for number systems are simple continued fractions, base-b expansions, Dedekind cuts and Cauchy sequences.

In [1] and [2], simple continued fractions and base-b expansions denote each number in number systems as a set of symbols. So both them denote number systems intuitively and join well in algebraical operations. In [3] and [4], Dedekind cuts and Cauchy sequences introduce infinite rational numbers to denote an irrational number. So both them denote number systems logically and join well in logical deduction.

In this note, first, it is proven that both simple continued fractions and base-b expansions fail to denote real numbers and thus lack logic; second, it is shown that Dedekind cuts and Cauchy sequences fail to intuitively join in algebraical operations and thus lack intuition.

However, mathematical logic has sufficiency of intuition and logic. In [9], formal language introduces producer "$\rightarrow$" to formalize intuitive language. In [10], propositional logic introduces connectives such as "$\neg$", "$\wedge$", "$\vee$", "$\Rightarrow$ / $\rightarrow$" and "$\Leftrightarrow$ / $\leftrightarrow$" to formalize logical deduction. Therefore, it is feasible to combine producer and connectives to deduce intuitive and logical notations for number systems.

The paper is organized as follows. In Section 2, we study the most common notation for number system —— decimals, and prove that they fail to denote real numbers. In Section 3, by comparing those common notations for number systems, we show that intuitive simple continued fractions and base-b expansions lack logic while logical Dedekind cuts and Cauchy sequences lack intuition. In Section 4, we construct a logical calculus and deduce numbers to intuitively and logically denote number systems.

[1]Department of Information Science and Communication, Nanjing University of Information Science and Technology, Nanjing, 210044, China.
[2]Current address: P.O.Box 383, Gulou Post Office, Gulou District, Nanjing, 210008, China.
E-mail address: pith_xie@ hotmail.com.
[*]Corresponding author.
[†]Received 22 December 2010; accepted 4 March 2011.

2. DECIMALS AND REAL NUMBER SYSTEM

In this section, we show a conceptual error in the proof to [1, THEOREM 134], and then correct [1, THEOREM 134].

Definition 2.1 A sequence $\{x_n\}$ in a metric space (X, d) is a convergent sequence if there exists a point $x \in X$ such that, for every $\epsilon > 0$, there exists an integer N such that $d(x, x_n) < \epsilon$ for every integer $n \geq N$. The point x is called the limit of the sequence $\{x_n\}$ and we write

$$x_n \rightarrow x \tag{2.1}$$

or

$$\lim_{n\to\infty} x_n = x. \tag{2.2}$$

Theorem 2.2 ([1, THEOREM 134]). *Any positive number ξ may be expressed as a decimal*

$$A_1 A_2 \cdots A_{s+1}.\, a_1 a_2 a_3 \cdots , \tag{2.3}$$

where $0 \leq A_1 < 10, 0 \leq A_2 < 10, \cdots, 0 \leq a_n < 10$*, not all A and a are 0, and an infinity of the a_n are less than 9. If $\xi \geq 1$, then $A_1 \geq 0$. There is a (1,1) correspondence between the numbers and the decimals, and*

$$\xi = A_1 \cdot 10^s + \cdots + A_{s+1} + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots . \tag{2.4}$$

Proof. Let $[\xi]$ be the integral part of ξ. Then we write

$$\xi = [\xi] + x = X + x, \tag{2.5}$$

where X is an integer and $0 \leq x < 1$, and consider X and x separately.

If $X > 0$ and $10^s \leq x < 10^{s+1}$, and A_1 and X_1 are the quotient and remainder when X is divided by 10^s, then $X = A_1 \cdot 10^s + X_1$, where $0 < A_1 = [10^{-s}X] < 10$, $0 \leq X_1 < 10^s$.

Similarly

$$\begin{array}{rcl} X_1 = & A_2 \cdot 10^{s-1} + X_2 & (0 \leq A_2 < 10, 0 \leq X_2 < 10^{s-1}), \\ X_2 = & A_3 \cdot 10^{s-2} + X_3 & (0 \leq A_3 < 10, 0 \leq X_3 < 10^{s-2}), \\ \cdots & \cdots & \cdots \\ X_{s-1} = & A_s \cdot 10 + X_s & (0 \leq A_s < 10, 0 \leq X_s < 10), \\ X_s = & A_{s+1} & (0 \leq A_{s+1} < 10). \end{array}$$

Thus X may be expressed uniquely in the form

$$X = A_1 \cdot 10^s + A_2 \cdot 10^{s-1} + \cdots + A_s \cdot 10 + A_{s+1}, \tag{2.6}$$

where every A is one of 0, 1, 2, $\cdots$, 9, and A_1 is not 0. We abbreviate this expression to

$$X = A_1 A_2 \cdots A_s A_{s+1}, \tag{2.7}$$

the ordinary representation of X in decimal notation.

Passing to x, we write

$$X = f_1 \quad (0 \leq f_1 < 1).$$

We suppose that $a_1 = [10f_1]$, so that

$$\frac{a_1}{10} \leq f_1 < \frac{a_1+1}{10};$$

a_1 is one of $0, 1, 2, \cdots, 9$, and

$$a_1 = [10f_1], \quad 10f_1 = a_1 + f_2 \quad (0 \leq f_2 < 1).$$

Similarly, we define $a_2, a_3, \cdots$ by

$$\begin{array}{lll} a_2 = [10f_2], & 10f_2 = a_2 + f_3 & (0 \leq f_3 < 1), \\ a_3 = [10f_3], & 10f_3 = a_3 + f_4 & (0 \leq f_4 < 1), \\ \cdots & \cdots & \cdots \end{array}$$

Every a_n is one of $0, 1, 2, \cdots, 9$. Thus

$$x = x_n + g_{n+1}, \tag{2.8}$$

where

$$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}, \tag{2.9}$$

$$0 \leq g_{n+1} = \frac{f_{n+1}}{10^n} < \frac{1}{10^n}. \tag{2.10}$$

We thus define a decimal $.a_1a_2a_3 \cdots a_n \cdots$ associated with x. We call $a_1, a_2, \cdots$ the first, second, $\cdots$ *digits* of the decimal.

Since $a_n < 10$, the series

$$\sum_1^{\infty} \frac{a_n}{10^n} \tag{2.11}$$

is convergent; and since $g_{n+1} \to 0$, its sum is x. We may therefore write

$$x = .\, a_1a_2a_3 \cdots, \tag{2.12}$$

the right-hand side being an abbreviation for the series (2.11).

If $f_{n+1} = 0$ for some n, *i.e.* if $10^n x$ is an integer, then

$$a_{n+1} = a_{n+2} = \cdots = 0.$$

In this case we say that the decimal *terminates*. Thus

$$\frac{17}{400} = .0425000 \cdots,$$

and we write simply $\frac{17}{400} = .0425$.

It is plain that the decimal for x will terminate if and only if x is a rational fraction whose denominator is of the form $2^\alpha 5^\beta$.

Since $\frac{a_{n+1}}{10^{n+1}} + \frac{a_{n+2}}{10^{n+2}} + \cdots = g_{n+1} < \frac{1}{10^n}$ and $\frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \cdots = \frac{9}{10^{n+1}(1-\frac{1}{10})} = \frac{1}{10^n}$, it is impossible that every a_n from a certain point on should be 9. With this reservation, every possible sequence (a_n) will arise from some x. We define x as the sum of the series (2.11), and x_n and g_{n+1} as in (2.8) and (2.9). Then $g_{n+1} < 10^{-n}$ for every n, and x yields the sequence required.

Finally, if

$$\sum_{1}^{\infty} \frac{a_n}{10^n} = \sum_{1}^{\infty} \frac{b_n}{10^n}, \tag{2.13}$$

and the b_n satisfy the conditions already imposed on the a_n, then $a_n = b_n$ for every n. For if not, let a_N and b_N be the first pair which differ, so that $|a_N - b_N| \geq 1$. Then

$$\left|\sum_{1}^{\infty} \frac{a_n}{10^n} - \sum_{1}^{\infty} \frac{b_n}{10^n}\right| \geq \frac{1}{10^N} - \sum_{N+1}^{\infty} \frac{|a_n - b_n|}{10^n} \geq \frac{1}{10^N} - \sum_{N+1}^{\infty} \frac{9}{10^n} = 0.$$

This contradicts (2.13) unless there is equality. If there is equality, then all of $a_{N+1}-b_{N+1}, a_{N+2}-b_{N+2}, \cdots$ must have the same sign and the absolute value 9. But then either $a_n = 9$ and $b_n = 0$ for $n > N$, or else $a_n = 0$ and $b_n = 9$, and we have seen that each of these alternatives is impossible. Hence $a_n = b_n$ for all n. In other words, different decimals correspond to different numbers.

We now combine (2.5), (2.7), and (2.12) in the form

$$\xi = X + x = A_1A_2 \cdots A_sA_{s+1}.\, a_1a_2a_3 \cdots ; \tag{2.14}$$

and the claim follows. □

According to Definition 2.1, the series (2.11) converges to the limit x. For an infinite sequence, however, its limit may not equal its $\omega - th$ number for any infinite number ω.

1. ω is a *transfinite cardinal number*[5]. Since the equalities and order on the fractions including transfinite cardinal numbers have not been defined, the equation $g_{\omega+1} = \frac{f_{\omega+1}}{10^\omega} = 0$ cannot be derived from given premises for any ω.

2. ω is an *infinite superreal number*[6] or an *infinite surreal number*[7]. Since the infinitesimal

$$g_{\omega+1} = \frac{f_{\omega+1}}{10^\omega} > 0$$

holds for every ω, the equation

$$g_{\omega+1} = \frac{f_{\omega+1}}{10^\omega} = 0$$

cannot be derived from given premises for any ω.

In summary, the equation $x = x_\omega + g_{\omega+1}$ cannot derives $x = x_\omega$ for any infinite number ω. Thus, (2.12) cannot be derived from given premises.

In fact, the proof to [1, THEOREM 134] confuses the limit and the $\omega - th$ number of the same infinite sequence for some infinite number ω.

According to the arguments above, we correct [1, THEOREM 134] as follows.

Theorem 2.3 *Any positive number ξ may be expressed as a limit of an infinite decimal sequence*

$$\lim_{n\to\infty} A_1A_2 \cdots A_{s+1}.\, a_1a_2a_3 \cdots a_n, \tag{2.15}$$

where $0 \leq A_1 < 10, 0 \leq A_2 < 10, \cdots, 0 \leq a_n < 10$*, not all A and a are 0, and an infinity of the a_n are less than 9. If $\xi \geq 1$, then $A_1 \geq 0$. There is a (1,1) correspondence between the numbers and the limits of infinite decimal sequences, and*

$$\xi = A_1 \cdot 10^s + \cdots + A_{s+1} + \lim_{n\to\infty} \sum \frac{a_n}{10^n}. \tag{2.16}$$

3. COMMON NOTATIONS FOR NUMBER SYSTEMS

3.1 Intuitive Notations

Simple continued fractions and base-b expansions construct intuitive symbols to denote number systems. They join well in algebraical operations and thus have sufficiency of intuition.

Definition 3.1 A finite continued fraction is a function

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_N}}}} \tag{3.1}$$

of $N + 1$ variables

$$a_0, a_1, \cdots, a_n, \cdots, a_N, \tag{3.2}$$

which is called finite simple continued fraction when $a_0, a_1, \cdots, a_N$ are integers such that $a_n > 0$ for all $n \geq 1$.

Finite simple continued fractions can be written in a compact abbreviated notation as

$$[a_0, a_1, a_2, \cdots, a_N]. \tag{3.3}$$

Definition 3.2 If $a_0, a_1, a_2, \cdots, a_n, \cdots$ is a sequence of integers such that $a_n > 0$ for all $n \geq 1$, then the notation

$$[a_0, a_1, a_2, \cdots] \tag{3.4}$$

denotes an infinite simple continued fraction.

Theorem 3.3 ([1, THEOREM 149]). *If p_n and q_n are defined by*

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2} \ (2 \leq n \leq N), \tag{3.5}$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (2 \leq n \leq N), \tag{3.6}$$

then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \tag{3.7}$$

Theorem 3.3 can be specialized for finite simple continued fractions as follows:

Theorem 3.4 $\{a_0, a_1, \cdots, a_n\}$ *is an integer sequence. If p_n and q_n are defined by*

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2} \ (2 \leq n \leq N), \tag{3.8}$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (2 \leq n \leq N), \tag{3.9}$$

then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \tag{3.10}$$

Theorem 3.4 can directly derive such a corollary as follows:

Corollary 3.5 *Any finite simple continued fraction can be represented by a rational number.*

Theorem 3.6 ([1, THEOREM 161]). *Any rational number can be represented by a finite simple continued fraction.*

According to Corollary 3.5 and [1, THEOREM 161], finite simple continued fractions are equivalent to rational numbers.

Theorem 3.7 ([1, THEOREM 161])**.** *Any rational number can be represented by a finite simple continued fraction.*

Theorem 3.8 ([1, THEOREM 170])**.** *Every irrational number can be expressed in just one way as an infinite simple continued fraction.*

Proof. We call

$$a'_n = [a_n, a_{n+1}, \cdots] \tag{3.11}$$

the n-th complete quotient of the continued fraction $x = [a_0, a_1, \cdots]$.

Clearly

$$\begin{aligned} a'_n &= \lim_{N\to\infty} [a_n, a_{n+1}, \cdots, a_N] \\ &= a_n + \lim_{N\to\infty} \frac{1}{[a_{n+1}, \cdots, a_N]} \\ &= a_n + \frac{1}{a'_{n+1}}, \end{aligned}$$

and in particular $x = a'_0 = a_0 + \frac{1}{a'_1}$.

Also $a'_n > a_n, a'_{n+1} > a_{n+1} > 0, 0 < \frac{1}{a'_{n+1}} < 1$; and so $a_n = [a'_n]$, the integral part of a'_n.

Let x be any real number, and let $a_0 = [x]$. Then

$$x = a_0 + \xi_0, \quad 0 \le \xi_0 < 1.$$

If $\xi_0 \neq 0$, we can write

$$\frac{1}{\xi_0} = a'_1, \quad [a'_1] = a_1, \quad a'_1 = a_1 + \xi_1, \quad 0 \le \xi_1 < 1.$$

If $\xi_1 \neq 0$, we can write

$$\frac{1}{\xi_1} = a'_2 = a_2 + \xi_2, \quad 0 \le \xi_2 < 1,$$

and so on. Also $a'_n = 1/\xi_{n-1} > 1$, and so $a_n \ge 1$, for $n \ge 1$. Thus,

$$x = [a_0, a'_1] = \left[a_0, a_1 + \frac{1}{a'_2}\right] = [a_0, a_1, a'_2] = [a_0, a_1, a_2, a'_3] = \cdots, \tag{3.12}$$

where $a_0, a_1, \cdots$ are integers and

$$a_1 > 0, \quad a_2 > 0, \cdots. \tag{3.13}$$

The system of equations

$$x = \quad a_0 + \xi_0 \quad (0 \le \xi_0 < 1),$$

$$\begin{aligned}
\frac{1}{\xi_0} = &\quad a'_1 = a_1 + \xi_1 \quad (0 \le \xi_1 < 1), \\
\frac{1}{\xi_1} = &\quad a'_2 = a_2 + \xi_2 \quad (0 \le \xi_2 < 1), \\
\cdots &\quad \cdots \quad \cdots
\end{aligned}$$

is known as the *continued fraction algorithm*. The algorithm continues so long as $\xi_n \neq 0$. If we eventually reach a value of n, say N, for which $\xi_n = 0$, the algorithm terminates and

$$x = [a_0, a_1, a_2, \cdots, a_N]. \tag{3.14}$$

In this case x is represented by a simple continued fraction, and is rational.

If x is an integer, then $\xi_0 = 0$ and $x = a_0$. If x is not integral, then

$$x = \frac{h}{k},$$

where h and k are integers and $k > 1$. Since

$$\frac{h}{k} = a_0 + \xi_0, \quad h = a_0 + \xi_0 k,$$

a_0 is the quotient, and $k_1 = \xi_0 k$ the remainder, when h is divided by k.

If $\xi_0 \neq 0$, then

$$a'_1 = \frac{1}{\xi_0} = \frac{k}{k_1} \tag{3.15}$$

and

$$\frac{k}{k_1} = a_1 + \xi_1, \quad k = a_1 k_1 + \xi_0 k_1;$$

thus a_1 is the quotient, and $k_2 = \xi_1 k_1$ the remainder, when k is divided by k_1. We thus obtain a series of equations

$$h = a_0 k + k_1, \quad k = a_1 k_1 + k_2, \quad k_1 = a_2 k_2 + k_3, \quad \cdots$$

continuing so long as $\xi_n \neq 0$, or, what is the same thing, so long as $k_{n+1} \neq 0$.

The non-negative integers $k, k_1, k_2, \cdots$ form a strictly decreasing sequence, and so $k_{N+1} = 0$ for some N. It follows that $\xi_N = 0$ for some N, and that the continued fraction algorithm terminates. This proves [1, THEOREM 161].

The system of equations

$$\begin{aligned}
h = &\quad a_0 k + k_1 \quad (0 < k_1 < k), \\
k = &\quad a_1 k_1 + k_2 \quad (0 < k_2 < k_1), \\
\cdots &\quad \cdots \quad \cdots \\
k_{N-2} = &\quad a_{N-1} k_{N-1} + k_N \quad (0 < k_N < k_{N-1}), \\
k_{N-1} = &\quad a_N k_N
\end{aligned}$$

is known as *Euclid's algorithm*.

If x is irrational the continued fraction algorithm cannot terminate. Hence it defines an infinite sequence of integers

$$a_0, a_1, a_2, \cdots, \tag{3.16}$$

and as before

$$x = [a_0, a'_1] = [a_0, a_1, a'_2] = \cdots = [a_0, a_1, a_2, \cdots, a_n, a'_{n+1}], \tag{3.17}$$

where $a'_{n+1} = a_{n+1} + \frac{1}{a'_{n+2}} > a_{n+1}$. Hence

$$x = a'_0 = \frac{a'_1 a_0 + 1}{a'_1} = \cdots = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}, \tag{3.18}$$

and so

$$x - \frac{p_n}{q_n} = \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n(a'_{n+1} q_n + q_{n-1})} = \frac{(-1)^n}{q_n(a'_{n+1} q_n + q_{n-1})}, \tag{3.19}$$

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n(a_{n+1} q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \leq \frac{1}{n(n+1)} \to 0, \tag{3.20}$$

when $n \to \infty$. Thus

$$x = \lim_{n\to\infty} \frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n, \cdots], \tag{3.21}$$

and the algorithm leads to the continued fraction whose value is x. □

In Section 2, we have proven that the limit of an infinite sequence may not equal the $\omega - th$ number of the same infinite sequence for any infinite number ω.

1. ω is a transfinite cardinal number. Since the equalities and order on the fractions including transfinite cardinal numbers have not been defined, the inequality $|x - \frac{p_\omega}{q_\omega}| < \frac{1}{\omega(\omega+1)}$ does not hold for any ω.

2. ω is an infinite superreal number or an infinite surreal number. Since the infinitesimal $\frac{1}{\omega(\omega+1)} > 0$ holds for every ω, the equation $|x - \frac{p_\omega}{q_\omega}| = 0$ or $x = \frac{p_\omega}{q_\omega}$ cannot be derived from given premises for any ω.

In summary, the inequality (3.20) cannot derives (3.21). In fact, (3.20) only derives $x = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [a_0, a_1, \cdots, a_n]$.

According to the arguments above, we correct [1, THEOREM 170] as follows.

Theorem 3.9 *Every irrational number can be expressed in just one way as a limit of an infinite simple continued fraction sequence.*

According to [2, §BF.2], we can define base-b expansions as follows:

Definition 3.10 Base-b expansion is an expression of number as follows.

$$c_n b^n + c_{n-1} b^{n-1} \cdots + c_2 b^2 + c_1 b^1 + c_0 b^0 + d_1 b^{-1} + d_2 b^{-2} \cdots + d_n b^{-n}, \tag{3.22}$$

where b represents the base, and c_i and d_i are place-value coefficients. The expansion would ordinarily be written without the plus signs and the powers of the base as follows:

$$c_n c_{n-1} \cdots c_2 c_1 c_0 \,.\, d_1 d_2 \cdots d_n, \tag{3.23}$$

where b^i is implied by the place-value property of the system.

According to 3.10, finite decimals are just base-10 expansions.

Definition 3.11 Base-variable expansions are base-b expansions for every finite integer b greater than 1.

Theorem 3.12 *Every base-variable expansion is equal to a rational number.*

Proof. According to Definition 3.11, every base-variable expansion x must also be a base-b expansion. Then

$$x = \pm a_n \cdots a_2 a_1 a_0 \,.\, a_{-1} a_{-2} \cdots a_{-n}. \tag{3.24}$$

According to Definition 3.10, it follows that

$$\pm a_n \cdots a_2 a_1 a_0 \,.\, a_{-1} a_{-2} \cdots a_{-n} = \pm \frac{\sum_{i=-n}^{n} a_i b^{i+n}}{b^n}. \tag{3.25}$$

Since both digit $0 \leq a_i < b$ and b are integers, $\pm \frac{\sum_{i=-n}^{n} a_i b^{i+n}}{b^n}$ must be a rational number. So the claim follows. □

Theorem 3.13 (The Fundamental Theorem of Arithmetic)**.** *Every natural number is either prime or can be uniquely factored as a product of primes in a unique way.*

Theorem 3.14 *Every base-b expansion for a constant b may be unequal to a rational number.*

Proof. According to the equation (3.25), every base-b expansion for a constant b may be expressed as follows:

$$x = \pm \frac{\sum_{i=-n}^{n} a_i b^{i+n}}{b^n}. \tag{3.26}$$

Since there exists infinite primes, there must exist a prime q such that $(q, b) = 1$. Since $q > 1$ and $b \neq 0$, it follows from Theorem 3.13 that for every $0 \leq a_i < b$ there exists

$$q \cdot \sum_{i=-n}^{n} a_i b^{i+n} \neq b^n. \tag{3.27}$$

Hence

$$\pm \frac{\sum_{i=-n}^{n} a_i b^{i+n}}{b^n} \neq \frac{1}{q}, \tag{3.28}$$

which holds for every $0 \leq a_i < b$. So the claim follows. □

From Theorem 3.12 and Theorem 3.14, we can conclude such a corollary as follows:

Corollary 3.15 *Base-variable expansions are included in rational numbers.*

According to the arguments above, no algorithms can determine the equalities between infinite simple continued fractions or infinite base-variable expansions and real numbers. As to the limits of infinite simple continued fraction sequences and those of infinite base-variable expansion sequences, they belong to logical notations and will be discussed in the next section.

In summary, both simple continued fractions and base-variable expansions lack logic and fail to denote real numbers.

3.2 Logical Notations

According to Definition 2.1, limit is based on infinite sequence. So the limits of infinite simple continued fraction sequences and those of infinite base-variable expansion sequences are also defined on infinite sequence.

In 1872, Dedekind and Cantor invented Dedekind cuts and Cauchy sequences respectively to denote number systems. However, both Dedekind cuts and Cauchy sequences are based on rational number system. In 1889, Peano published a study giving an axiomatic approach to the natural numbers[8]. Peano Axioms can also be extended to define rational numbers. Then both Dedekind cuts and Cauchy sequences join well in logical deduction and thus have sufficiency of logic.

In nature, Dedekind cuts and Cauchy sequences introduce infinite rational numbers to denote an irrational number. In Dedekind cuts, an irrational cut (A, B) is defined on two infinite rational sets A and B. In Cauchy sequences, an irrational number is defined on an equivalence class of some infinite rational sequence.

Although it is feasible to logically define algebraical operations on infinite sets or infinite sequences, it is impossible to intuitively execute these infinite algebraical operations in a finite period. So the limits of infinite simple continued fraction sequences and those of infinite base-variable expansion sequences lack intuition and fail to join in algebraical operations. For the same reason, both Dedekind cuts and Cauchy sequences lack intuition and fail to join in algebraical operations.

4. LOGICAL CALCULUS

Simple continued fractions and base-variable expansions fail to denote real numbers, while the limits of infinite simple continued fraction sequences, the limits of infinite base-variable expansion sequences, logical Dedekind cuts and Cauchy sequences fail to join in algebraical operations. In mathematical logic, logical calculus is a formal system to abstract and analyze the induction and deduction apart from specific meanings. In this section, however, we construct a logical calculus by virtue of formal language and deduce numbers to intuitively and logically denote number systems. The logical calculus not only denotes real numbers, but also allows them to join in algebraical operations.

The introduction of formal language aims to use computer fast execute real number operations. For clarity, we will explain the logical calculus with natural language.

In [9], the producer "$\to$" substitutes the right permutations for the left permutations to produce new permutations. In [10], the connectives "$\neg$", "$\wedge$", "$\vee$", "$\Rightarrow$" and "$\Leftrightarrow$" stand for "not", "and", "or", "implies" and "if and only if" respectively. Here, the producer "$\to$" is considered as a predicate symbol and embedded into logical calculus.

Definition 4.1 $\{\Phi, \Psi\}$ is a logical calculus such that:

$$\Phi\{ \tag{4.1}$$

$$V\{\emptyset, a, b\cdots\}, \tag{4.2}$$

$$C\{\emptyset, 1, +\cdots\}, \tag{4.3}$$

$$P\{\emptyset, \in, \subseteq, \to, |, =, <\cdots\}, \tag{4.4}$$

$$V\circ C\{\emptyset, a, b\cdots, 1, +\cdots, aa, ab\cdots, a1, a+\cdots, ba, bb\cdots, b1, b+\cdots, \tag{4.5}$$

$$aaa, aab\cdots, aa1, aa+\cdots, baa, bab\cdots, ba1, ba+\cdots\},$$

$$C\circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\}, \tag{4.6}$$

$$V\circ C\circ P\{\emptyset, a, b\cdots, 1, +\cdots, \in, \subseteq\cdots, aa, ab\cdots, a1, a+\cdots, a\in, a\subseteq\cdots, \tag{4.7}$$

$$ba, bb\cdots, b1, b+\cdots, b\in, b\subseteq\cdots, aaa, aab\cdots, aa1, aa+\cdots, aa\in, aa\subseteq\cdots,$$

$$baa, bab\cdots, ba1, ba+\cdots, ba\in, ba\subseteq\cdots\},$$

$$(\hat{a}\in V)\Leftrightarrow((\hat{a}\equiv\emptyset)\vee(\hat{a}\equiv a)\vee(\hat{a}\equiv b)\cdots), \tag{4.8}$$

$$(\hat{a}\in C)\Leftrightarrow((\hat{a}\equiv\emptyset)\vee(\hat{a}\equiv 1)\vee(\hat{a}\equiv +)\cdots), \tag{4.9}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots), \quad (4.10)$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots), \quad (4.11)$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots), \quad (4.12)$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C)) \cdots \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)) \cdots, \quad (4.13)$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \quad (4.14)$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))) \cdots$$

},

$$\Psi\{ \quad (4.15)$$

$$(\bar{a} \subseteq \bar{b}) \Leftrightarrow (\bar{b} = \bar{c}\bar{a}\bar{d}), \quad (4.16)$$

$$(\bar{a} \to \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \Rightarrow (\bar{a} \to \bar{b}\bar{e}\bar{d}), \quad (4.17)$$

$$(\bar{\bar{a}} \to \bar{\bar{b}}|\bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{b}}) \wedge (\bar{\bar{a}} \to \bar{\bar{c}})), \quad (4.18)$$

$$(\bar{\bar{a}}|\bar{\bar{b}} \to \bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{c}}) \wedge (\bar{\bar{b}} \to \bar{\bar{c}})), \quad (4.19)$$

$$(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{b} < \bar{a}), \quad (4.20)$$

$$(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{a} = \bar{b}), \quad (4.21)$$

$$(\bar{a} < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow (\bar{a} < \bar{c}), \quad (4.22)$$

$$(\bar{a} < \bar{b}) \wedge (\bar{a} \in (C \circ C)) \wedge (\bar{b} \in (C \circ C)) \Rightarrow (\bar{a} \wedge \bar{b}), \quad (4.23)$$

$$(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}), \quad (4.24)$$

$$(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} = \bar{e}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}), \quad (4.25)$$

$$(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}), \quad (4.26)$$

$$(\bar{a} < \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \to \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} < \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}), \quad (4.27)$$

$$(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} \to \bar{e}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}\}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}), \quad (4.28)$$

$$(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} < \bar{d}\bar{f}\bar{e}), \quad (4.29)$$

$$(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} < \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}), \quad (4.30)$$

$$(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c}\bar{f}\bar{d} < \bar{e}), \quad (4.31)$$

$$(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c}\bar{g}\bar{d} < \bar{e}\bar{g}\bar{f}), \quad (4.32)$$

$$(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \to \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} < \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}), \quad (4.33)$$

$$\bar{a} = \bar{a}, \tag{4.34}$$
$$(\bar{a} = \bar{b}) \Rightarrow (\bar{b} = \bar{a}), \tag{4.35}$$
$$(\bar{a} = \bar{b}) \Rightarrow \neg(\bar{a} < \bar{b}), \tag{4.36}$$
$$(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}), \tag{4.37}$$
$$(\bar{a}\bar{b}\bar{c}) \wedge (\bar{b} = \bar{d}) \Rightarrow (\bar{a}\bar{b}\bar{c} = \bar{a}\bar{d}\bar{c}), \tag{4.38}$$
$$(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \rightarrow \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}), \tag{4.39}$$
$$(\bar{a} = \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \rightarrow \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} = \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}), \tag{4.40}$$
$$(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \rightarrow \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} = \bar{d}\bar{f}\bar{e}), \tag{4.41}$$
$$(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \rightarrow \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} = \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}), \tag{4.42}$$
$$(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} = \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \rightarrow \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} = \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}) \tag{4.43}$$
$$\}.$$

First, we will explain the primitive symbols of the logical calculus $\{\Phi, \Psi\}$ with natural language.

The symbols "{", "}", ",", "(", ")" are punctuation. The symbol "$\emptyset$" indicates emptiness. The symbol "$\cdots$" indicates an omission.

(4.1) denotes Φ as a set of notations and particular axioms between { and }. Different logical calculus correspond to different notations and particular axioms.

(4.2) denotes V as a set of variables between { and }.

(4.3) denotes C as a set of constants between { and }.

(4.4) denotes P as a set of predicate symbols between { and }.

(4.5) denotes $V \circ C$ as a set of concatenations between V and C.

(4.6) denotes $C \circ C$ as a set of concatenations between C and C.

(4.7) denotes $V \circ C \circ P$ as a set of concatenations among V, C and P.

(4.8) $\sim$ (4.12) define a set of axioms on the binary predicate symbol $\in$.

(4.13) defines an axiom on new variables ranging over $V \circ C$.

(4.14) defines an axiom on the binary predicate symbol $\subseteq$.

(4.15) denotes Ψ as a set of general axioms between { and }. Different logical calculus correspond to the same general axioms.

(4.16) defines an axiom on the binary predicate symbol $\subseteq$.

(4.17) defines an axiom on the binary predicate symbol $\rightarrow$.

(4.18) $\sim$ (4.19) define a set of axioms on the binary predicate symbol $|$.

(4.20) $\sim$ (4.33) define a set of axioms on the binary predicate symbol $<$.

(4.34) $\sim$ (4.43) define a set of axioms on the binary predicate symbol $=$.

Then, we will prove that the logical calculus $\{\Phi, \Psi\}$ can deduce common number systems.

Definition 4.2 In a logical calculus $\{\Phi, \Psi\}$, if $\bar{a} \equiv true$, then $\bar{a}$ is a number.

Theorem 4.3

If $\Phi\{$
$$V\{\emptyset, a, b\}, \tag{4.44}$$
$$C\{\emptyset, 1, +\}, \tag{4.45}$$

$$P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <\}, \tag{4.46}$$

$$V \circ C\{\emptyset, a, b\cdots, 1, +\cdots, aa, ab\cdots, a1, a+\cdots, ba, bb\cdots, b1, b+\cdots, aaa, aab\cdots, aa1, aa+\cdots, baa, bab\cdots, ba1, ba+\cdots\}, \tag{4.47}$$

$$C \circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\}, \tag{4.48}$$

$$V \circ C \circ P\{\emptyset, a, b\cdots, 1, +\cdots, \in, \subseteq\cdots, aa, ab\cdots, a1, a+\cdots, a\in, a\subseteq\cdots, ba, bb\cdots, b1, b+\cdots, b\in, b\subseteq\cdots, aaa, aab\cdots, aa1, aa+\cdots, aa\in, aa\subseteq\cdots, baa, bab\cdots, ba1, ba+\cdots, ba\in, ba\subseteq\cdots\}, \tag{4.49}$$

$$(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots), \tag{4.50}$$

$$(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots), \tag{4.51}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv 1)\cdots \vee (\hat{a} \equiv aa) \vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a1)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa1)\cdots), \tag{4.52}$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+)\cdots \vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+)\cdots), \tag{4.53}$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv\in)\cdots \vee (\hat{a} \equiv aa) \vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a\in)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa\in)\cdots), \tag{4.54}$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C)) \wedge(\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)), \tag{4.55}$$

$$\begin{aligned}
&((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})\\
&\vee(\bar{a} \subseteq \bar{g}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})\\
&\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})\\
&\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge\\
&((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})\\
&\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),
\end{aligned} \tag{4.56}$$

$$a \rightarrow 1|1+a, \tag{4.57}$$

$$a < 1+a, \tag{4.58}$$

$$\bar{a} \wedge \bar{b} \Rightarrow (\bar{a} + \bar{b} = \bar{b} + \bar{a}) \tag{4.59}$$

},

then $N\{\Phi, \Psi\}$ *denotes natural number system.*

Proof.

$(A1)$	$(a \rightarrow 1\|1+a) \Rightarrow (a \rightarrow 1)$	*by* (4.57), (4.18)
$(A2)$	$\Rightarrow (a \rightarrow 1+a)$	*by* (4.19)
$(A3)$	$(a < 1+a) \Rightarrow (1 < 1+1)$	*by* (4.58), $(A1)$, (4.29)
$(A4)$	$\Rightarrow 1$	*by* (4.23)

$$\begin{array}{lll}
(A5) & \Rightarrow (1+1) & by\ (A3),(4.23)\\
(A6) & \Rightarrow (1+a<1+1+a) & by\ (4.58),(A2),(4.29)\\
(A7) & \Rightarrow (1+1<1+1+1) & by\ (A1),(4.29)\\
(A8) & \Rightarrow (1+1) & by\ (4.23)\\
(A9) & \Rightarrow (1+1+1) & by\ (A7),(4.23)\\
\vdots & \vdots & \vdots
\end{array}$$

Then we deduce the numbers from $N\{\Phi,\Psi\}$:

$$1, 1+1, 1+1+1, 1+1+1+1\cdots$$

$$\begin{array}{lll}
(B1) & 1+1=1+1 & by\ (A4),(4.59)\\
(B2) & 1+1+1=1+1+1 & by\ (A4),(A5),(4.59)\\
(B3) & 1+1+1+1=1+1+1+1 & by\ (A5),(4.59)\\
\vdots & \vdots & \vdots
\end{array}$$

Then we deduce the equalities on deducible numbers from $N\{\Phi,\Psi\}$:

$$1+1=1+1, 1+1+1=1+1+1, 1+1+1+1=1+1+1+1\cdots$$

The deducible numbers correspond to the natural numbers as follows:

$$\begin{array}{rcl}
1 & \equiv & 1,\\
1+1 & \equiv & 2,\\
1+1+1 & \equiv & 3,\\
\vdots & \vdots & \vdots
\end{array}$$

The equalities on deducible numbers correspond to the addition in natural number system. So the claim follows. □

Theorem 4.4

$$\begin{aligned}
&If\ \Phi\{\\
&V\{\emptyset,a,b,c\}, && (4.60)\\
&C\{\emptyset,1,+,[,],-\}, && (4.61)\\
&P\{\emptyset,\in,\subseteq,\to,|,=,<\}, && (4.62)\\
&V\circ C\{\emptyset,a,b\cdots,1,+\cdots,aa,ab\cdots,a1,a+\cdots,ba,bb\cdots,b1,b+\cdots, && (4.63)\\
&aaa,aab\cdots,aa1,aa+\cdots,baa,bab\cdots,ba1,ba+\cdots\},\\
&C\circ C\{\emptyset,1,+\cdots,11,1+\cdots,111,11+\cdots\}, && (4.64)\\
&V\circ C\circ P\{\emptyset,a,b\cdots,1,+\cdots,\in,\subseteq\cdots,aa,ab\cdots,a1,a+\cdots,a\in,a\subseteq\cdots, && (4.65)\\
&ba,bb\cdots,b1,b+\cdots,b\in,b\subseteq\cdots,aaa,aab\cdots,aa1,aa+\cdots,aa\in,aa\subseteq\cdots,\\
&baa,bab\cdots,ba1,ba+\cdots,ba\in,ba\subseteq\cdots\},\\
&(\hat{a}\in V)\Leftrightarrow((\hat{a}\equiv\emptyset)\vee(\hat{a}\equiv a)\vee(\hat{a}\equiv b)\cdots), && (4.66)\\
&(\hat{a}\in C)\Leftrightarrow((\hat{a}\equiv\emptyset)\vee(\hat{a}\equiv 1)\vee(\hat{a}\equiv +)\cdots), && (4.67)
\end{aligned}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \tag{4.68}$$
$$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots),$$
$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \tag{4.69}$$
$$\vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots),$$
$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \tag{4.70}$$
$$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots),$$
$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \tag{4.71}$$
$$\wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C))$$
$$\wedge(\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)),$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \tag{4.72}$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$
$$\vee(\bar{a} \subseteq \bar{g}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$
$$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$
$$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$$
$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$
$$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$$
$$a \to 1|[aba], \tag{4.73}$$
$$b|c \to +|-, \tag{4.74}$$
$$a < [1 + a], \tag{4.75}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}b\bar{b}c\bar{c}] = [[\bar{a}b\bar{b}]c\bar{c}]), \tag{4.76}$$
$$\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1 - 1]), \tag{4.77}$$
$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}]), \tag{4.78}$$
$$\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a}), \tag{4.79}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - \bar{b} + \bar{c}] = [\bar{a} + \bar{c} - \bar{b}]), \tag{4.80}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}]), \tag{4.81}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}]), \tag{4.82}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]), \tag{4.83}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}] \tag{4.84}$$
$$\},$$

then $Z\{\Phi, \Psi\}$ *denotes integral number system.*

Proof.

$(A1)$	$(a \to 1\|[aba]) \Rightarrow (a \to 1)$	*by* (4.73), (4.18)
$(A2)$	$\Rightarrow (a \to [aba])$	*by* (4.73), (4.18)
$(A3)$	$\Rightarrow (a \to [1b1])$	*by* $(A2), (A1)$, (4.17)
$(A4)$	$(b\|c \to +\|-) \Rightarrow (b\|c \to -)$	*by* (4.74), (4.18)

$$
\begin{array}{lcl}
(A5) & \Rightarrow (b \rightarrow -) & by\ (4.19) \\
(A6) & \Rightarrow (a \rightarrow [1-1]) & by\ (A3),(A5),(4.17) \\
(A7) & (a < [1+a]) \Rightarrow (1 < [1+1]) & by\ (4.75),(A1),(4.29) \\
(A8) & \Rightarrow 1 & by\ (4.23) \\
(A9) & \Rightarrow [1+1] & by\ (A7),(4.23) \\
(A10) & (a < [1+a]) \Rightarrow ([1-1] < [1+[1-1]]) & by\ (4.75),(A6),(4.29) \\
(A11) & \Rightarrow ([1-1] < [[1-1]+1]) & by\ (4.78),(4.24) \\
(A12) & \Rightarrow ([1-1] < 1) & by\ (4.79),(4.24) \\
(A13) & \Rightarrow [1-1] & by\ (4.23) \\
\vdots & \vdots & \vdots
\end{array}
$$

Then we deduce the numbers from $Z\{\Phi, \Psi\}$:

$$[1-1], 1, [1-1-1], [1+1], [1+1+1]\cdots$$

$$
\begin{array}{lcl}
(B1) & [1+1] = [1+1] & by\ (A8),(4.78) \\
(B2) & [1+[1+1]] = [[1+1]+1] & by\ (A8),(A9),(4.78) \\
(B3) & [1+[1-1]] = [[1-1]+1] & by\ (A8),(A13),(4.78) \\
(B4) & [[1+1]+[1-1]] = [[1-1]+[1+1]] & by\ (A9),(A13),(4.78) \\
\vdots & \vdots & \vdots
\end{array}
$$

Then we deduce the equalities on deducible numbers from $Z\{\Phi, \Psi\}$:

$$[1+[1+1]] = [[1+1]+1], [1+[1-1]] = [[1-1]+1]\cdots$$

The deducible numbers correspond to the integral numbers as follows:

$$
\begin{array}{rcl}
\vdots & \vdots & \vdots, \\
[1-1-[1+1]] & \equiv & -2, \\
[1-1-1] & \equiv & -1, \\
[1-1] & \equiv & 0, \\
1 & \equiv & 1, \\
[1+1] & \equiv & 2, \\
[1+1+1] & \equiv & 3, \\
[1+1+1+1] & \equiv & 4, \\
\vdots & \vdots & \vdots.
\end{array}
$$

The equalities on deducible numbers correspond to the addition and subtraction in integral number system. So the claim follows. □

Theorem 4.5

If $\Phi\{$

$$V\{\emptyset, a, b, c, d\}, \tag{4.85}$$

$$C\{\emptyset, 1, +, [,], -\}, \tag{4.86}$$

$$P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <\}, \tag{4.87}$$

$$V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a + \cdots, ba, bb \cdots, b1, b + \cdots, \\ aaa, aab \cdots, aa1, aa + \cdots, baa, bab \cdots, ba1, ba + \cdots\}, \tag{4.88}$$

$$C \circ C\{\emptyset, 1, + \cdots, 11, 1 + \cdots, 111, 11 + \cdots\}, \tag{4.89}$$

$$V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a + \cdots, a \in, a \subseteq \cdots, \\ ba, bb \cdots, b1, b + \cdots, b \in, b \subseteq \cdots, aaa, aab \cdots, aa1, aa + \cdots, aa \in, aa \subseteq \cdots, \\ baa, bab \cdots, ba1, ba + \cdots, ba \in, ba \subseteq \cdots\}, \tag{4.90}$$

$$(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots), \tag{4.91}$$

$$(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots), \tag{4.92}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \\ \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots), \tag{4.93}$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \\ \vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots), \tag{4.94}$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \\ \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots), \tag{4.95}$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \\ \wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge \\ (\bar{j} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)), \tag{4.96}$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee \\ (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))), \tag{4.97}$$

$$a \rightarrow 1|[aba], \tag{4.98}$$

$$b \rightarrow +|-, \tag{4.99}$$

$$c|d \rightarrow b| + +| - -, \tag{4.100}$$

$$a < [1 + a], \tag{4.101}$$

$$(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow ([\bar{a} + \bar{c}] < [\bar{b} + \bar{c}]), \tag{4.102}$$

$$(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow ([\bar{c} - \bar{b}] < [\bar{c} - \bar{a}]), \tag{4.103}$$

$$([1 - 1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a} - -\bar{c}] < [\bar{b} - -\bar{c}]), \tag{4.104}$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}\bar{c}\bar{b}d\bar{c}] = [[\bar{a}\bar{c}\bar{b}]d\bar{c}]), \tag{4.105}$$

$$\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1 - 1]), \tag{4.106}$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}]), \tag{4.107}$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a}), \tag{4.108}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - \bar{b} + \bar{c}] = [\bar{a} + \bar{c} - \bar{b}]), \tag{4.109}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}]), \tag{4.110}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}]), \tag{4.111}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]), \tag{4.112}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}]), \tag{4.113}$$
$$\bar{a} \Rightarrow ([\bar{a} + +1] = \bar{a}), \tag{4.114}$$
$$\neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{a}] = 1), \tag{4.115}$$
$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + +\bar{b}] = [\bar{b} + +\bar{a}]), \tag{4.116}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + \bar{c}]] = [[\bar{a} + +\bar{b}] + [\bar{a} + +\bar{c}]]), \tag{4.117}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]]), \tag{4.118}$$
$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + +\bar{c}]] = [[\bar{a} + +\bar{b}] + +\bar{c}]), \tag{4.119}$$
$$\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{b} + +\bar{b}] = \bar{a}), \tag{4.120}$$
$$\bar{a} \wedge \bar{b} \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} - -\bar{c} + +\bar{b}] = [\bar{a} + +\bar{b} - -\bar{c}]) \wedge ([[\bar{a} + \bar{b}] - -\bar{c}] = [[\bar{a} - -\bar{c}] + [\bar{b} - -\bar{c}]]) \wedge ([[\bar{a} - \bar{b}] - -\bar{c}] = [[\bar{a} - -\bar{c}] - [\bar{b} - -\bar{c}]]) \wedge ([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}])), \tag{4.121}$$
$$\bar{a} \wedge \neg([\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]]) \Rightarrow (([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}])) \tag{4.122}$$
},

then $Q\{\Phi, \Psi\}$ *denotes rational number system.*

Proof.

$(A1)$	$(a \to 1\|[aba]) \Rightarrow (a \to 1)$	*by* (4.98), (4.18)
$(A2)$	$\Rightarrow (a \to [aba])$	*by* (4.98), (4.18)
$(A3)$	$\Rightarrow (a \to [[aba]ba])$	*by* $(A2)$, (4.17)
$(A4)$	$\Rightarrow (a \to [[1 - 1] - 1])$	*by* $(A1)$, (4.99)
$(A5)$	$(a < [1 + a]) \Rightarrow (1 < [1 + 1])$	*by* (4.101), $(A1)$, (4.29)
$(A6)$	$\Rightarrow 1$	*by* (4.23)
$(A7)$	$\Rightarrow [1 + 1]$	*by* $(A5)$, (4.23)
$(A8)$	$\Rightarrow ([aba] < [1 + [aba]])$	*by* $(A5)$, $(A2)$, (4.29)
$(A9)$	$\Rightarrow ([1 + 1] < [1 + [1 + 1]])$	*by* $(A1)$, (4.99), (4.29)
$(A10)$	$\Rightarrow ([1 + 1] < [[1 + 1] + 1])$	*by* (4.107), (4.24)
$(A11)$	$\Rightarrow ([1 + 1] < [1 + 1 + 1])$	*by* (4.105)
$(A12)$	$\Rightarrow ([1 - 1] < [1 + [1 - 1]])$	*by* $(A8)$, $(A1)$, (4.99), (4.29)
$(A13)$	$\Rightarrow ([1 - 1] < [[1 - 1] + 1])$	*by* (4.107), (4.24)
$(A14)$	$\Rightarrow ([1 - 1] < 1)$	*by* (4.108)
$(A15)$	$\Rightarrow ([1 - 1] < [1 + 1 + 1])$	*by* $(A14)$, $(A5)$, $(A11)$, (4.22)
$(A16)$	$\Rightarrow ([1 - -[1 + 1 + 1]] < [[1 + 1] - -[1 + 1 + 1]])$	*by* $(A14)$, $(A5)$, $(A15)$, (4.104)
$(A17)$	$\Rightarrow [1 - -[1 + 1 + 1]]$	*by* (4.23)
$(A18)$	$\Rightarrow [[1 + 1] - -[1 + 1 + 1]]$	*by* $(A16)$, (4.23)
$\vdots$	$\vdots$	$\vdots$

Then we deduce the numbers from $Q\{\Phi, \Psi\}$:

$$[1-1], [[1-1]-[1--[1+1]]], [1--[1+1]], 1\cdots$$

$$\begin{array}{lccr} (B1) & [1+[1--[1+1+1]]] = [[1--[1+1+1]]+1] & by\ (A6), (A17), (4.107) \\ (B2) & [1++[1--[1+1+1]]] = [[1--[1+1+1]]++1] & by\ (A6), (A17), (4.116) \\ \vdots & \vdots & \vdots \end{array}$$

Then we deduce the equalities on deducible numbers from $Q\{\Phi, \Psi\}$:

$$[1+1] = [1+1], [1++[1--[1+1+1]]] = [[1--[1+1+1]]++1]\cdots$$

The deducible numbers correspond to the rational numbers as follows:

$$\begin{array}{rcl} \vdots & \vdots & \vdots, \\ [1-1-1] & \equiv & -1, \\ \vdots & \vdots & \vdots, \\ [1-1-[1--[1+1]]] & \equiv & -\frac{1}{2}, \\ \vdots & \vdots & \vdots, \\ [1-1] & \equiv & 0, \\ \vdots & \vdots & \vdots, \\ [1-[1--[1+1]]] & \equiv & \frac{1}{2}, \\ \vdots & \vdots & \vdots, \\ 1 & \equiv & 1, \\ \vdots & \vdots & \vdots, \\ [1+[1--[1+1]]] & \equiv & \frac{3}{2}, \\ \vdots & \vdots & \vdots, \\ [1+1] & \equiv & 2, \\ \vdots & \vdots & \vdots, \\ [1+1+[1--[1+1]]] & \equiv & \frac{5}{2}, \\ \vdots & \vdots & \vdots, \\ [1+1+1] & \equiv & 3, \\ \vdots & \vdots & \vdots. \end{array}$$

The equalities on deducible numbers correspond to the addition, subtraction, multiplication, division in rational number system. So the claim follows. □

Definition 4.6 Real number system is a logical calculus $R\{\Phi, \Psi\}$ such that:

$$\Phi\{$$

$$V\{\emptyset, a, b, c, d, e, f, g, h, i, j, k, l\}, \tag{4.123}$$

$$C\{\emptyset, 1, +, [,], -, /, \top, \bot, \_\}, \tag{4.124}$$

$$P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \|\}, \tag{4.125}$$

$$V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a + \cdots, ba, bb \cdots, b1, b + \cdots, \tag{4.126}$$

$$aaa, aab \cdots, aa1, aa + \cdots, baa, bab \cdots, ba1, ba + \cdots\},$$

$$C \circ C\{\emptyset, 1, + \cdots, 11, 1 + \cdots, 111, 11 + \cdots\}, \tag{4.127}$$

$$V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a + \cdots, a \in, a \subseteq \cdots, \tag{4.128}$$

$$ba, bb \cdots, b1, b + \cdots, b \in, b \subseteq \cdots, aaa, aab \cdots, aa1, aa + \cdots, aa \in, aa \subseteq \cdots,$$

$$baa, bab \cdots, ba1, ba + \cdots, ba \in, ba \subseteq \cdots\},$$

$$(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots), \tag{4.129}$$

$$(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots), \tag{4.130}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \tag{4.131}$$

$$\vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots),$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \tag{4.132}$$

$$\vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots),$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \tag{4.133}$$

$$\vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots),$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \tag{4.134}$$

$$\wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge$$

$$(\bar{j} \in (V \circ C)) \wedge (\bar{k} \in (V \circ C)) \wedge (\bar{l} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge$$

$$(\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)),$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \tag{4.135}$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$

$$\vee (\bar{a} \subseteq \bar{g}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$

$$\vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$$

$$\vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$$

$$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}, \bar{k}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$$

$$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}) \vee (\bar{a} \subseteq \bar{k}))),$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}, \bar{k}, \bar{l}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$$

$$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}) \vee (\bar{a} \subseteq \bar{k}) \vee (\bar{a} \subseteq \bar{l}))),$$

$$(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{f}\bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \wedge \neg(\bar{f} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{g}\}) \wedge \tag{4.136}$$

$$((\bar{b} \rightarrow \bar{h}) \| (\bar{f} \rightarrow \bar{i})) \Rightarrow (\bar{a}\bar{h}\bar{c} = \bar{d}\bar{h}\bar{e}\bar{i}\bar{g}),$$

$$(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e} = \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \wedge \neg(\bar{d} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}\}) \wedge ((\bar{b} \rightarrow \bar{g}) \| (\bar{d} \rightarrow \bar{h})) \tag{4.137}$$

$$\Rightarrow (\bar{a}\bar{g}\bar{c}\bar{h}\bar{e} = \bar{f}),$$

$$a \to 1|[aba], \quad (4.138)$$

$$b \to +|-, \quad (4.139)$$

$$c|d \to e|f|g, \quad (4.140)$$

$$e \to +| + e, \quad (4.141)$$

$$f \to -| - f, \quad (4.142)$$

$$g \to /|/g, \quad (4.143)$$

$$(h \to +)||(i \to -), \quad (4.144)$$

$$(h \to +h)||(i \to -i), \quad (4.145)$$

$$(i \to -)||(h \to +), \quad (4.146)$$

$$(i \to -i)||(h \to +h), \quad (4.147)$$

$$(h \to +)||(j \to /), \quad (4.148)$$

$$(h \to +h)||(j \to /j), \quad (4.149)$$

$$k \to [1+1]|[1+k], \quad (4.150)$$

$$l \to 1|[1+l], \quad (4.151)$$

$$a < [1+a], \quad (4.152)$$

$$(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a}+\bar{c}] < [\bar{b}+\bar{c}]) \wedge ([\bar{c}-\bar{b}] < [\bar{c}-\bar{a}])), \quad (4.153)$$

$$([1-1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow (([\bar{a}--\bar{c}] < [\bar{b}--\bar{c}]) \wedge ([\bar{c}--\bar{b}] < [\bar{c}--\bar{a}])), \quad (4.154)$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow (1 < [\bar{a}--f\bar{b}]), \quad (4.155)$$

$$(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \Rightarrow (1 < [\bar{b}/g\bar{a}]), \quad (4.156)$$

$$(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge (1 < \bar{c}) \Rightarrow ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]), \quad (4.157)$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \wedge (1 < \bar{c}) \wedge ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]) \Rightarrow (\bar{a} < \bar{b}), \quad (4.158)$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]), \quad (4.159)$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \wedge (1 < \bar{c}) \wedge ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]) \Rightarrow (\bar{b} < \bar{c}), \quad (4.160)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}c\bar{b}d\bar{c}] = [[\bar{a}c\bar{b}]d\bar{c}]), \quad (4.161)$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}-\bar{b}] = [\bar{a}/\bar{b}]), \quad (4.162)$$

$$\bar{a} \wedge \neg(\bar{b} = [1-1]) \Rightarrow ([\bar{a}--\bar{b}] = [\bar{a}//\bar{b}]), \quad (4.163)$$

$$\bar{a} \Rightarrow ([\bar{a}-\bar{a}] = [1-1]), \quad (4.164)$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}+\bar{b}] = [\bar{b}+\bar{a}]), \quad (4.165)$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a}-\bar{b}]+\bar{b}] = \bar{a}), \quad (4.166)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}-\bar{b}+\bar{c}] = [\bar{a}+\bar{c}-\bar{b}]), \quad (4.167)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}+\bar{c}]] = [[\bar{a}+\bar{b}]+\bar{c}]), \quad (4.168)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}-\bar{c}]] = [[\bar{a}+\bar{b}]-\bar{c}]), \quad (4.169)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}-[\bar{b}+\bar{c}]] = [[\bar{a}-\bar{b}]-\bar{c}]), \quad (4.170)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}-[\bar{b}-\bar{c}]] = [[\bar{a}-\bar{b}]+\bar{c}]), \quad (4.171)$$

$$\bar{a} \Rightarrow ([\bar{a}++1] = \bar{a}), \quad (4.172)$$

$$\neg(\bar{a} = [1-1]) \Rightarrow ([\bar{a}--\bar{a}] = 1), \quad (4.173)$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}++\bar{b}] = [\bar{b}++\bar{a}]), \quad (4.174)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}++[\bar{b}++\bar{c}]] = [[\bar{a}++\bar{b}]++\bar{c}]), \quad (4.175)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}++[\bar{b}+\bar{c}]] = [[\bar{a}++\bar{b}]+[\bar{a}++\bar{c}]]), \quad (4.176)$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]]), \tag{4.177}$$

$$\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([[\bar{a} - -\bar{b}] + +\bar{b}] = \bar{a}), \tag{4.178}$$

$$\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow (([\bar{a} - -\bar{b} + +\bar{c}] = [\bar{a} + +\bar{c} - -\bar{b}]) \wedge ([[\bar{a} + \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] + [\bar{c} - -\bar{b}]]) \wedge ([[\bar{a} - \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] - [\bar{c} - -\bar{b}]])), \tag{4.179}$$

$$\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}])), \tag{4.180}$$

$$\bar{a} \Rightarrow ([\bar{a} + + + 1] = \bar{a}), \tag{4.181}$$

$$\bar{a} \Rightarrow ([1 + + + \bar{a}] = 1), \tag{4.182}$$

$$\neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} + + + [1 - 1]] = 1), \tag{4.183}$$

$$([1 - 1] < \bar{a}) \Rightarrow ([[1 - 1] + + + \bar{a}] = [1 - 1]), \tag{4.184}$$

$$([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow (([\bar{a} - - - \bar{b} + + + \bar{b}] = \bar{a}) \wedge ([\bar{a} - - - \bar{b} + + + \bar{c}] = [\bar{a} + + + \bar{c} - - - \bar{b}]) \wedge ([\bar{a} + + + [\bar{c} - -\bar{b}]] = [[\bar{a} + + + \bar{c}] - - - \bar{b}])), \tag{4.185}$$

$$([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a} + + + [\bar{b}///\bar{a}]] = \bar{b}) \wedge ([[\bar{a} + + + \bar{c}]///\bar{b}] = [\bar{c} + +[\bar{a}///\bar{b}]]) \wedge ([[\bar{a} - -\bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - -[\bar{b} + + + \bar{c}]])), \tag{4.186}$$

$$([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ((([[\bar{a}///\bar{c}] - -[\bar{b}///\bar{c}]] = [\bar{a}///\bar{b}]) \wedge ([[\bar{a} + +\bar{b}]///\bar{c}] = [[\bar{a}///\bar{c}] + [\bar{b}///\bar{c}]]) \wedge ([[\bar{a} - -\bar{b}]///\bar{c}] = [[\bar{a}///\bar{c}] - [\bar{b}///\bar{c}]])), \tag{4.187}$$

$$\neg(\bar{a} = [1 - 1]) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([\bar{a} + +\bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] + +[\bar{b} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} + +\bar{c}]] = [[\bar{a} + + + \bar{b}] + + + \bar{c}]) \wedge([\bar{a} + + + [\bar{b} + \bar{c}]] = [[\bar{a} + + + \bar{b}] + +[\bar{a} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} - \bar{c}]] = [[\bar{a} + + + \bar{b}] - -[\bar{a} + + + \bar{c}]])), \tag{4.188}$$

$$([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} - - - [\bar{b} + +\bar{c}]] = [[\bar{a} - - - \bar{b}] - - - \bar{c}]) \wedge ([\bar{a} - - - [\bar{b} - -\bar{c}]] = [[\bar{a} - - - \bar{b}] + + + \bar{c}])), \tag{4.189}$$

$$([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([[\bar{a} + +\bar{b}] - - - \bar{c}] = [[\bar{a} - - - \bar{c}] + +[\bar{b} - - - \bar{c}]]) \wedge ([[\bar{a} - -\bar{b}] - - - \bar{c}] = [[\bar{a} - - - \bar{c}] - -[\bar{b} - - - \bar{c}]])), \tag{4.190}$$

$$(1 < \bar{a}) \Rightarrow ([\bar{a} + e1] = \bar{a}), \tag{4.191}$$

$$(1 < \bar{a}) \Rightarrow ([\bar{a} - f1] = \bar{a}), \tag{4.192}$$

$$(1 < \bar{a}) \Rightarrow ([1 + +e\bar{a}] = 1), \tag{4.193}$$

$$(1 < \bar{a}) \Rightarrow ([1 - -f\bar{a}] = 1), \tag{4.194}$$

$$(1 < \bar{a}) \Rightarrow ([1///\bar{a}] = [1 - 1]), \tag{4.195}$$

$$(1 < \bar{a}) \Rightarrow ([\bar{a}/g\bar{a}] = 1), \tag{4.196}$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{a}i\bar{b}]h\bar{b}] = \bar{a}), \tag{4.197}$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{a}h\bar{b}]i\bar{b}] = \bar{a}), \tag{4.198}$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{b}h[\bar{a}j\bar{b}]] = \bar{a}), \tag{4.199}$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{b}h\bar{a}]j\bar{b}] = \bar{a}), \tag{4.200}$$

$$(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{a} + e\bar{b}] = [\bar{a}e[\bar{a} + e[\bar{b} - 1]]]), \tag{4.201}$$

$$\top\_1\_1\_ = [1 - 1], \tag{4.202}$$

$$\top\_1\_[1+1]\_ = 1, \tag{4.203}$$
$$\bot\_1\_1\_ = 1, \tag{4.204}$$
$$\bot\_1\_[1+1]\_ = 1, \tag{4.205}$$
$$\top\_k\_[[[1+1]++l]-1]\_ = \top\_[k-1]\_l\_, \tag{4.206}$$
$$\bot\_k\_[[[1+1]++l]-1]\_ = \bot\_[k-1]\_l\_, \tag{4.207}$$
$$\top\_k\_[[1+1]++l]\_ = [\top\_[k-1]\_l\_ + \top\_[k-1]\_[l+1]\_], \tag{4.208}$$
$$\bot\_k\_[[1+1]++l]\_ = [\bot\_[k-1]\_l\_ + \bot\_[k-1]\_[l+1]\_], \tag{4.209}$$
$$(1 < \bar{a}) \wedge (\top\_\bar{b}\_\bar{c}\_) \wedge (\bot\_\bar{b}\_\bar{c}\_) \Rightarrow ([\bar{a} + h[\top\_\bar{b}\_\bar{c}\_ - -\bot\_\bar{b}\_\bar{c}\_]] = \tag{4.210}$$
$$[[\bar{a} + h\top\_\bar{b}\_\bar{c}\_] - i\bot\_\bar{b}\_\bar{c}\_])$$
$$\}.$$

It should be noted that (4.202) ~ (4.210) restrict $[\top\_\bar{b}\_\bar{c}\_ - -\bot\_\bar{b}\_\bar{c}\_]$ to be a Farey fraction. In the following, we will deduce some numbers and equalities as examples.

$(A1)$	$(a \to 1\|[aba]) \Rightarrow (a \to 1)$	*by* (4.138), (4.18)
$(A2)$	$\Rightarrow (a \to [aba])$	*by* (4.138), (4.18)
$(A3)$	$\Rightarrow (a \to [[aba]ba])$	*by* $(A2)$, (4.17)
$(A4)$	$\Rightarrow (a \to [[1-1]-1])$	*by* $(A1)$, (4.139)
$(A5)$	$(a < [1+a]) \Rightarrow (1 < [1+1])$	*by* (4.152), $(A1)$, (4.29)
$(A6)$	$\Rightarrow 1$	*by* (4.23)
$(A7)$	$\Rightarrow [1+1]$	*by* $(A5)$, (4.23)
$(A8)$	$\Rightarrow ([aba] < [1+[aba]])$	*by* $(A5)$, $(A2)$, (4.29)
$(A9)$	$\Rightarrow ([1+1] < [1+[1+1]])$	*by* $(A1)$, (4.139), (4.29)
$(A10)$	$\Rightarrow ([1+1] < [[1+1]+1])$	*by* (4.165), (4.24)
$(A11)$	$\Rightarrow ([1+1] < [1+1+1])$	*by* (4.161)
$(A12)$	$\Rightarrow ([1-1] < [1+[1-1]])$	*by* $(A8)$, $(A1)$, (4.139), (4.29)
$(A13)$	$\Rightarrow ([1-1] < [[1-1]+1])$	*by* (4.165), (4.24)
$(A14)$	$\Rightarrow ([1-1] < [1-1+1])$	*by* (4.161)
$(A15)$	$\Rightarrow ([1-1] < 1)$	*by* (4.166)
$(A16)$	$\Rightarrow ([1-1] < [1+1+1])$	*by* $(A15)$, $(A5)$, $(A11)$, (4.22)
$(A17)$	$\Rightarrow [1-1]$	*by* $(A15)$, (4.23)
$(A18)$	$\Rightarrow [1+1+1]$	*by* $(A16)$, (4.23)
$(A19)$	$\Rightarrow ([1-1] < [1+1])$	*by* $(A15)$, $(A5)$, (4.22)
$(A20)$	$([[1+1]--[1+1]] < [[1+1+1]--[1+1]])$	*by* $(A5)$, $(A11)$, (4.154)
$(A21)$	$\Rightarrow (1 < [[1+1+1]--[1+1]])$	*by* $(A20)$, (4.173)
$(A22)$	$\Rightarrow [[1+1+1]--[1+1]]$	*by* $(A21)$, (4.23)
$\vdots$	$\vdots$	$\vdots$

Then we deduce the numbers from $R\{\Phi, \Psi\}$:

$$[1-1], [[1-1]-[[1+1]----[1+1]]], [[1+1+1]----[1+1]] \cdots$$

$$(B1) \quad (\top\_k\_[[1+1]++l]\_ = [\top\_[k-1]\_l\_ + \top\_[k-1]\_[l+1]\_]) \quad by\ (4.208)$$

($B2$) $\Rightarrow (\top\_[1+1]\_[[1+1]++l]\_ = [\top\_[[1+1]-1]\_l\_+ \top\_[[1+1]-1]\_[l+1]\_])$ *by* (4.150), (4.18), (4.42)

($B3$) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[[1+1]-1]\_1\_+ \top\_[[1+1]-1]\_[1+1]\_])$ *by* (4.151), (4.18), (4.42)

($B4$) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[1+1-1]\_1\_+ \top\_[1+1-1]\_[1+1]\_])$ *by* (4.161)

($B5$) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[1-1+1]\_1\_+ \top\_[1-1+1]\_[1+1]\_])$ *by* (4.167)

($B6$) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[[1-1]+1]\_1\_+ \top\_[[1-1]+1]\_[1+1]\_])$ *by* (4.161)

($B7$) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_1\_1\_ + \top\_1\_[1+1]\_])$ *by* (4.166)

($B8$) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = [\top\_1\_1\_ + \top\_1\_[1+1]\_])$ *by* (4.172)

($B9$) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = [[1-1]+1])$ *by* (4.202), (4.203)

($B10$) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = 1)$ *by* (4.166)

($B11$) $\Rightarrow ([1-1] < \top\_[1+1]\_[1+1]\_)$ *by* ($A15$), (4.24)

($B12$) $\Rightarrow (\top\_[1+1]\_[1+1]\_)$ *by* (4.23)

($B13$) $(\bot\_k\_[[1+1]++l]\_ = [\bot\_[k-1]\_l\_ + \bot\_[k-1]\_[l+1]\_])$ *by* (4.209)

($B14$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++l]\_ = [\bot\_[[1+1]-1]\_l\_+ \bot\_[[1+1]-1]\_[l+1]\_])$ *by* (4.150), (4.18), (4.42)

($B15$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[[1+1]-1]\_1\_+ \bot\_[[1+1]-1]\_[1+1]\_])$ *by* (4.151), (4.18), (4.42)

($B16$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[1+1-1]\_1\_+ \bot\_[1+1-1]\_[1+1]\_])$ *by* (4.161)

($B17$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[1-1+1]\_1\_+ \bot\_[1-1+1]\_[1+1]\_])$ *by* (4.167)

($B18$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[[1-1]+1]\_1\_+ \bot\_[[1-1]+1]\_[1+1]\_])$ *by* (4.161)

($B19$) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_1\_1\_ + \bot\_1\_[1+1]\_])$ *by* (4.166)

($B20$) $\Rightarrow (\bot\_[1+1]\_[1+1]\_ = [\bot\_1\_1\_ + \bot\_1\_[1+1]\_])$ *by* (4.172)

($B21$) $\Rightarrow (\bot\_[1+1]\_[1+1]\_ = [1+1])$ *by* (4.204), (4.205)

($B22$) $\Rightarrow ([1-1] < \bot\_[1+1]\_[1+1]\_)$ *by* ($A19$), (4.24)

($B23$) $\Rightarrow (\bot\_[1+1]\_[1+1]\_)$ *by* (4.23)

($B24$) $([[1+1]+e[[1+1+1]--[1+1]]] = [[1+1]e[[1+1]+e[[[1+1+1]--[1+1]]-1]]])$ *by* ($A5$), ($A21$), (4.201)

($B25$) $\Rightarrow ([[1+1]++e[[1+1+1]--[1+1]]] = [[1+1]+e[[1+1]++e[[[1+1+1]--[1+1]]-1]]])$ *by* (4.141)

(B26) $\Rightarrow ([[1+1]+++e[[1+1+1]--[1+1]]] =$
$[[1+1]++e[[1+1]+++e[[[1+1+1]--[1+1]]-1]]])$ *by* (4.141)

(B27) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[[1+1+1]--[1+1]]-1]]])$ *by* (4.141)

(B28) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[[[1+1]+1]--[1+1]]-1]]])$ *by* (4.161)

(B29) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++$
$[[[[1+1]--[1+1]]+[1--[1+1]]]-1]]])$ *by*(4.179)

(B30) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[1+[1--[1+1]]]-1]]])$ *by* (A19), (4.173)

(B31) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[[1--[1+1]]+1]-1]]])$ *by* (4.165)

(B32) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[1--[1+1]]+1-1]]])$ *by* (4.161)

(B33) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[1--[1+1]]-1+1]]])$ *by* (4.167)

(B34) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[[[1--[1+1]]-1]+1]]])$ *by* (4.161)

(B35) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]++++[1--[1+1]]]])$ *by* (4.166)

(B36) $([[1+1]+h[\top\_[1+1]\_[1+1]\_--\bot\_[1+1]\_[1+1]\_]] =$
$[[[1+1]+h\top\_[1+1]\_[1+1]\_]-i\bot\_[1+1]\_[1+1]\_])$ *by* (A5), (B12), (B23), (4.210)

(B37) $\Rightarrow ([[1+1]+h[1--[1+1]]] = [[[1+1]+h1]-i[1+1]])$ *by* (B10), (B21), (4.41)

(B38) $\Rightarrow ([[1+1]++h[1--[1+1]]] =$
$[[[1+1]++h1]--i[1+1]])$ *by* (4.136), (4.145)

(B39) $\Rightarrow ([[1+1]+++h[1--[1+1]]] =$
$[[[1+1]+++h1]---i[1+1]])$ *by* (4.136), (4.145)

(B40) $\Rightarrow ([[1+1]++++[1--[1+1]]] =$
$[[[1+1]++++1]----[1+1]])$ *by* (4.136), (4.144)

(B41) $([[1+1]+e1] = [1+1])$ *by* (A5), (4.191)

(B42) $\Rightarrow ([[1+1]++e1] = [1+1])$ *by* (4.141)

(B43) $\Rightarrow ([[1+1]+++e1] = [1+1])$ *by* (4.141)

(B44) $\Rightarrow ([[1+1]++++1] = [1+1])$ *by* (4.141)

(B45) $\Rightarrow ([[1+1]++++[1--[1+1]]] = [[1+1]----[1+1]])$ *by* (B40), (B44), (4.37)

(B46) $\Rightarrow ([[1+1]++++[[1+1+1]--[1+1]]] =$
$[[1+1]+++[[1+1]----[1+1]]])$ *by*(B35), (B45), (4.37)

$\vdots \qquad \vdots \qquad \vdots$

Then we deduce the equalities on deducible numbers from $R\{\Phi, \Psi\}$:

$$[[1+1]++[[1+1]---[1+1]]] = [[[1+1]---[1+1]]++[1+1]],$$
$$[[1+1]++++[[1+1+1]--[1+1]]] = [[1+1]+++[[1+1]----[1+1]]]$$
$$\vdots$$

The deducible numbers correspond to the real numbers as follows:

$$\begin{array}{rcl}
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+1+1]----[1+1]]] & \equiv & \\
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+1]---[1+1]]] & \equiv & -\sqrt[2]{2}, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+1]---[1+1+1]]] & \equiv & -\sqrt[3]{2}, \\
\vdots & \vdots & \vdots, \\
[[1-1]-1] & \equiv & -1, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[1--[1+1]]] & \equiv & -\frac{1}{2}, \\
\vdots & \vdots & \vdots, \\
[1-1] & \equiv & 0, \\
\vdots & \vdots & \vdots, \\
[1-[1--[1+1]]] & \equiv & \frac{1}{2}, \\
\vdots & \vdots & \vdots, \\
[[1+1]///[1+1+1]] & \equiv & \log_3 2, \\
\vdots & \vdots & \vdots, \\
1 & \equiv & 1, \\
\vdots & \vdots & \vdots, \\
[[1+1]---[1+1+1]] & \equiv & \sqrt[3]{2}, \\
\vdots & \vdots & \vdots, \\
[[1+1]---[1+1]] & \equiv & \sqrt[2]{2}, \\
\vdots & \vdots & \vdots, \\
[1+[1--[1+1]]] & \equiv & \frac{3}{2}, \\
\vdots & \vdots & \vdots, \\
[[1+1+1]///[1+1]] & \equiv & \log_2 3, \\
\vdots & \vdots & \vdots,
\end{array}$$

$$
\begin{array}{rcl}
[[1+1+1]----[1+1]] & \equiv & \\
\vdots & \vdots & \vdots, \\
[1+1] & \equiv & 2, \\
\vdots & \vdots & \vdots, \\
[1+1+[1--[1+1]]] & \equiv & \frac{5}{2}, \\
\vdots & \vdots & \vdots, \\
[1+1+1] & \equiv & 3, \\
\vdots & \vdots & \vdots, \\
[[[1+1+1]----[1+1]]++[1+1]] & \equiv & \\
\vdots & \vdots & \vdots.
\end{array}
$$

The equalities on deducible numbers correspond to addition, subtraction, multiplication, division, power operation and more other operations in real number system. Note that in the correspondence above some irrational numbers such as $[[1+1+1]----[1+1]]$, $[[[1+1+1]----[1+1]]++[1+1]]$ and $[[1-1]-[[1+1+1]----[1+1]]]$ do not correspond to any irrational number based on traditional operations, which however can be constructed by the logical calculus $R\{\Phi,\Psi\}$.

Then the logical calculus $R\{\Phi,\Psi\}$ not only derives the irrational numbers, but also makes its deducible numbers join in algebraical operations. So the logical calculus $R\{\Phi,\Psi\}$ intuitively and logically denote real number system.

5. ACKNOWLEDGEMENTS

The author wishes to thank those anonymous referees for valuable and constructive suggestions and comments which lead to a considerable improvement in this article.

REFERENCES

[1] Hardy, G. H., & Wright, E. M.(1979). *An Introduction to the Theory of Numbers*(5th edition). Oxford: Oxford University Press.

[2] Bender, E. A., & Williamson, S. G.(2005). *A Short Course in Discrete Mathematics*. San Diego: Dover Publications.

[3] Rudin, Walter.(1976). *Principles of Mathematical Analysis*(3rd edition). New York: McGraw-Hill.

[4] Katz, V. J.(2004). *A History of Mathematics*(brief edition). New Jersey: Pearson Education.

[5] Halmos, P. R.(1974). *Naive Set Theory*. New York: Springer-Verlag.

[6] Dales, H. G., & Woodin, W. H.(1996). *Super-Real Fields*. Oxford: Clarendon Press.

[7] Conway, J. H.(1976). *On Numbers and Games*. New York: Academic Press.

[8] Enderton, H. B.(1977). *Elements of Set Theory*. New York: Academic Press.

[9] Linz, P.(2001). *An Introduction to Formal Languages and Automata*(3rd edition). Sudbury: Jones and Bartlett Publishers.

[10] Nerode, A., & Shore, R. A.(1997). *Logic for Application*(2nd edition). New York: Springer-Verlag.

[11] Enderton, H. B.(2001). *A Mathematical Introduction to Logic*(2nd edition). New York: Academic Press.

[12] Ebbinghaus, H. D., Flum, J., & Thomas, W.(1994). *Mathematical Logic* (2nd edition). New York: Springer-Verlag.

[13] Haros, C.(1802). Tables pour evaluer une fraction ordinaire avec autand de decimals qu'on voudra; et pour trover la fraction ordinaire la plus simple, et qui a approche sensiblement d'une fraction decimale. *J. Ecole Polytechn, 4*, 364-368.