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Decentralized Model for a Two-Stage Supply Chain With Exogenous Demand

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Abstract

This paper presents a formal mathematical model for managing problems of stochastic demands; confronting many industries in the society today. We consider a two- stage supply chain where the upstream manufacturer (stage2) must always fill exogenous demands from the downstream manufacturer (stage1) using two types of expediting : Overtime production and outsourcing.

Key words

Supply chain management(SCM); Supply chain(SC); Inventory; Backorder; Optimality

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1. INTRODUCTION

A supply chain, typically, consists of suppliers, manufacturing centre, warehouses, distribution centre and retailers as well as raw materials, work-in-process inventory and finished products that flow between the facilities. In traditional supply chain situations, downstream facilities make decisions about their order quantities without regard to the actual inventory available upstream. If the upstream facilities do not have enough inventories on hand to fill the orders, it is often assumed that the downstream facility will take what it can get and backorder the rest or outsource elsewhere. In order to avoid these shortages, the upstream facilities have traditionally set their inventory levels high enough so that the likelihood of not meeting downstream demand is low. Every manufacturer usually maintains a reasonable inventory of goods to ensure smooth operations. Inventory is a necessary evil-too little of it causes costly interruptions, too much results in idle capital. The inventory problem determines the inventory level that balances the two extreme cases. An important factor in the formulation and solution of an inventory model is that the demand (per unit time) of an item may be deterministic (known with certainty) or stochastic (described by a probability distribution).

2. REVIEW OF RELATED LITERATURE

The term supply chain management arose in the late 1980s and came into widespread use in the 1990s. Prior to that time, businesses used terms such as "logistics" and "operations management" instead. The supply chain management literature offers many variations on the same theme. The most common definition [Houlihan (1985), Stevens (1989), Lee and Billington (1993), and Lamming (1996)] is a *system of suppliers, manufacturers, distributors, retailers, and customers where materials flow downstream from suppliers to customers, and information flows in both directions*.

The year 1958 may be considered the inception of stochastic inventory control, with the publication of studies in the mathematical theory of inventory and production Arrow, K., J, Karlin and Scarf, H. (1958). Almost all current articles in the field can trace their ideas back to this excellent text.The entire supply chain success is dependent on a good inventory system.

Lately, several articles are continually being published in this area. M.E. Seliaman and Ab Rahman Amad (2008) developed a model that dealt with different inventory coordination mechanisms between the chain members in a three-stage, non-serial supply chain system. Their assumption is that demand is stochastic at the retailer's end. Pablo, A., Miranda, Rodrigo, A.G. (2009) proposed a sequential heuristic approach to optimize inventory service levels in a two-stage supply chain. Their proposed approach deals with service level and inventory decisions; simultaneously, with network design decisions and incorporates unfulfilled demand costs in a previous inventory location model.

R.M. Hill, M.Seifbarghy and D.K. Smith(2007) considered a single-item two-echelon, continuous-review inventory model where a number of retailers have their stock replenished from a central warehouse. The warehouse in turn replenishes stock from an external supplier. S.S. Alireza, M.E. Kurz and J. Ashayeri(2010) addressed specific inventory management decisions with transportation cost consideration in a multi-level environment. They developed two models-namely decentralized ordering and centralized ordering model to investigate the effect of collective ordering by retailers on the total inventory cost of the system.

Xueipng Li*, Yuerong Chen (2010), studied a single-product inventory system which involves a supplier, a retailer, and differentiated customers. Inventory control, in recent times, is contracted to a vendor. Vendor-managed inventory (VMI) is emerging as a significant development in the trend towards collaboration and information sharing in supply chain management. Biredra, K.M., Sirinivsan, R.(2004) provide a new explanation for the reasons retailers might be interested in VMI. Optimal policies for a capacitated two-stage inventory system was investigated by Rodney,P.P, and Roma, K.(2004). Their paper demonstrates optimal policies for capacitated serial multi-echelon production/inventory systems. Discrete-time inventory model with stochastic demands with a constant lead time and lost sales was considered by Paul Zipkins(2008).

With standard assumptions of single-location system, linear production costs, holding costs, penalty cost and full backlogging, a base-stock policy is optimal for linear-ordering case (or no ordering cost), (Karlin,1958), an (s, S) policy is optimal for the linear-plus-fixed-ordering cost case (or a fixed ordering/setup cost), Scarf(1960); Iglehart,(1963), and an (R, nQ) policy is optimal for the batch ordering case (Veinott, 1965). Detailed results of an (s, S) policy explicitly for discrete demand can be found in Veinott and Wagner (1965), and Zheng and Federgruen (1991). Scarf proved that in general, (s, S) policies are optimal for inventory control problems with setup costs for production. Veinott proved same result. Both authors considered inventory problems over a finite horizon. Zheng generalized the results of Scarf and Veinott in his article over the infinite horizon in a novel way.

Yun Zhou, Xiaobo Zhao (2010) worked on periodic review inventory system that serves two demand classes with different priorities. Unsatisfied demands in the high-priority class are lost, whereas those in the low-priority class are backlogged. They formulated the problem as a dynamic programming model and characterize the structure of the optimal replenishment policy. B.Q. Rieksts, éA.Ventura(2010) paper discusses inventory models over an infinite planning horizon with constant demand rate and two modes of transportation.

In this work, we consider problems over an infinite horizon hence we rely heavily on the results from Zheng. However, one area where our assumptions differ from Zheng's are that he assumes that backorders are allowed; we modify this assumption in line with Eric, L.Huggins (2002, 2007). Huggins considered a problem with stochastic demand where the downstream facility's supply requests are always met by the upstream facility and backordering are not allowed at the upstream stage, hence the need for expediting using overtime production and premium freight. We shall, again digress away from Huggins in work. We replace his premium freight with outsourcing and add the value of information sharing into our models of the two-stage supply chains.

3. STATEMENT OF THE PROBLEM

We consider a periodic review two-stage supply chain situation where an upstream manufacturer (stage2) must always meet the supply requests from the downstream supplier (stage1). If we assume each period to be a day, for convenience, such that each day stage1 and stage2, each, produces up to chosen inventory levels, and at the end of the day, stage1places an exogenous demand on stage2 for raw materials. If stage2 cannot meet this demand from the current inventory and regular production then there exists a shortage which must be filled using overtime production. At the end of overtime production, the part that could not be produced is outsourced.

4. MATHEMATICAL FORMULATIONS

Our ultimate goal in this research is to find the optimal policy that minimizes the expected total discounted cost over the infinite horizon for the two - stage supply chain management. We define the variables of our models as follows:
Indices: k, indexes discrete time., π, stationary policies, $\pi = \{\mu, \mu, \ldots\}$.
Parameters
α: discount factor
Π: set of all admissible policies π
$p_i(\cdot)$: the probability that an order will be placed
X_K: state of the system at time k and summarizes the past information needed for future optimization.
Z_k: the control variable to be selected at time k Y_k: Random parameter also called disturbance. D_k: Exogenous demand N: The horizon or number of times control is applied.
$r(X_k)$: Penalty cost for holding stock.
cZ_k: the unit cost of ordering Z_k.
Our stock shall evolve according to the discrete-time equation:

$$X_{k+1} = X_k + Z_k - D_k \tag{1}$$

for all $k, X_k \in S, Z_k \in C, Y_k \in D$ where S and C are non empty sets and D is a countable set.
The cost per stage $g : S \times C \times D \mapsto \mathbb{R}$ is given, and defined as:

$$g(X_k, Z_k, Y_k) = r(X_k) + cZ_k. \tag{2}$$

We denote by Π, the set of all admissible policies , that is, the set of all sequences of functions

$$\pi = \{\mu_0, \mu_1, \ldots\}, \text{ where } \mu_k : S \mapsto C,\ \mu_k(X_k) \in Z(X_k)\ \forall\ X_k \in S,\ k \geq 0$$

have identical statistics and are characterized by probabilities $P(\cdot \mid X_k, Z_k)$ defined on D, where $P(Y_k|X_k, Z_k)$ is the probability of the occurrence of Y_k, when the current state and control are X_k and Z_k respectively, but not on values of prior disturbances $Y_{k-1}, 1, \ldots, Y_0$. We define our stationary policy as an admissible policy of the form $\pi = \{\mu, \mu, \ldots\}$ and its corresponding cost function is denoted by J We shall refer to $\{\mu, \mu, \ldots\}$ as the stationary policy μ. Hence we say that μ is optimal if

$$J_\mu(X) = J^*(X) \ \forall \text{ states } X \tag{3}$$

Finally, we minimize the optimal cost function over an infinite horizon as follows:

$$\text{Minimize } \lim_{N\to\infty} E_{D_k}\{\sum_{k=0}^{N-1} \alpha^k g(X_k, Z_k, Y_k)\}. \tag{4}$$

$$\text{Subject to } g(X_k, Z_k, Y_k) \geq 0 \ \forall \ (X_k, Z_k, Y_k) \in S \times C \times D$$

$$Z_k \geq 0, \ k \geq 0, \ldots, N-1$$

$$X_k \subset S, \ Z_k \subset C, \ Y_k \subset D \text{ and } g : S \times C \times D \mapsto \mathbb{R} \text{ is the cost per stage,}$$

where D is a countable set, control Z_k is constraint to take values in a given nonempty subset $Z(X_k) \in C$ which depend on the current state $X_k[Z_k \in Z(X_k) \ \forall \ X_k \in S], Z_k, \ k \geq 0$ to determine the optimal inventory control policies.

5. RESEARCH METHODOLOGY

The argument that minimizes equation (4) above is our main interest. This is, typically, an inventory control problem. To solve this problem, we shall use the principles of optimality in dynamic programming developed by Bellman(1957) and follow the notational conventions of Bertsekas(1995).

6. OPTIMALITY CONDITION: (BELLMANS' EQUATION)

An optimal policy has the property that whatever the initial state and the initial decisions, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.
Proposition 1 The optimal cost function J^* satisfies

$$J^*_\pi(X_0) = \min_{\pi\in\Pi} E_D\{g(X_k, Z_k, D_k) + \alpha J^*_\pi[f(X_k, Z_k, D_k)]\}, \ \forall \ X_k \in S.$$

Or equivalently, $J^* = TJ^*$. Furthermore, J^* is the unique solution of this equation within the class of bounded functions.
Corollary 1 For every stationary policy μ, the associated cost function satisfies

$$J_\mu(x) = E_D\{g(X, Z, D) + \alpha J_\mu(f(X, Z, D))\} \ \forall \ X \in S.$$

Or equivalently $J_\mu = T_\mu J_\mu$.
And J_μ is the unique solution of this equation within the class of bounded functions.
Proposition 2 Under either assumption P or N, the optimal cost function, $J^*_\mu(X_0)$ satisfies

$$J^*_\pi(X_0) = \min_{\pi\in\prod} E_D\{g(X_k, Z_k, D_k) + \alpha J^*_\pi[f(X_k, Z_k, D_k)]\}. \tag{5}$$

Or equivalently, $J^* = TJ^*$.
Corollary 2 Let μ be a stationary policy. Then under assumptions P or N, we've

$$J_\mu = E_D\{g(X_k, Z_k, D_k) + \alpha J^*_\pi[f(X_k, Z_k, D)]\}, \ X_k \in S.$$

Or equivalently, $J_\mu = T_\mu J_\mu$.

Proposition 3 A stationary policy μ is optimal if and only if $\mu(x)$ attains the minimum in corollary 2 for each $x \in S$, that is $TJ^* = T_\mu J^*$.

Proposition 4 For every state X_0, the optimal cost $J^*(X_0)$ of the basic problem is equal to $J_0(X_0)$, where the function J_0 is given by the last step of the following algorithm, which proceeds backward in time from period $N-1$ to period 0:

$$J_N(X_N) = g_N(X_N),$$

$$J_k(X_k) = \min_{Z_k \in U_k(X_k)} E_D\{g_k(X_k, Z_k, D) + J_{k+1}(f_k(X_k, Z_k, D))\},\ k = 0, 1, 2, \dots, N-1$$

where, the expectation is taken with respect to the probability distribution of D which depends on X_k and Z_k. Furthermore, if $Z_k^* = \mu_k^*(X_k)$ minimizes the right hand side of equation above for each X_k and k, the policy $\Pi^* = \{\mu_0^*, \dots, \mu_{N-1}^*\}$ is optimal.

The argument of the preceding proof provides an interpretation of $J_k(X_k)$ as the optimal cost for an $(N-K)$-stage problem starting at state X_K and time K, and ending at time N. We consequently, call $J_k(X_k)$ the cost-to-go at state X_k and time k, and refer to J_k as the cost-to-go function at time k.

Consider a k-stage policy $\pi = \{\mu_0, \mu_1, \dots, \mu_{K-1}\}$. Then the expression $(T_{\mu_0}, T_{\mu_1}, \dots, T_{\mu_{k-1}} J)(x)$ is defined recursively for $i = 0, \dots, k-2$ by

$$(T_{\mu_k}, T_{\mu_{k+1}}, \dots, T_{\mu_{k-1}} J)(x) = (T_{\mu_k}(T_{\mu_{k+1}}, \dots, T_{\mu_{k-1}} J)(x)$$

and represents the cost of the policy π for the k-stage, α discounted problem with initial state X, cost per stage g and terminal cost function $\alpha^k J$.

$$J_\mu(X) = \lim_{N\to\infty} (T_\mu^N J)(X)\ \forall\ X \in S.$$

6.0 Backward recursion for the two- stage supply chain management problem Here, $K = 2$, we write out this algorithm for stage1 and stage2 as follows:

$$\begin{aligned}
(T^2 J)(x) &= \min_{\pi\in\Pi} E_D\{g(X, Z, D) + \alpha(TJ)(f(X, Z, D))\} \\
&= \min_{\pi\in\Pi} E_{D_1}\{g(X_1, Z_1, D_1)\} + \alpha \min_{\pi\in\Pi} E_{D_2}\{g(f(X_1, Z_1, D_1), Z_2, D_2) \\
&\quad +\alpha J(f(f(X_1, Z_1, D_1), Z_2, D_2))\} \\
&= \min_{\pi\in\Pi} E_{D_1}\{g(X_1, Z_1, D_1)\} + \min_{\pi\in\Pi} E_{D_2}\{\alpha g(f(f(X_1, Z_1, D_1), Z_2, D_2))\}.
\end{aligned}$$

We now present the two - stage supply chain management activities as follows: Define the following variables:

D_t: The exogenous demand experienced by stage1 during period t, usually a day (for convenience).
$X_{1,t}$: Stage1 inventory level at the start of period t
$Y_{1,t}$: Stage1 production quantity during period
$Z_{1,t}$: Stage1 inventory position after production during period t
$X_{2,t}$: Stage2 inventory level at the beginning of period t
$Y_{2,t}$: Stage2 regular production quantity during period t.
$Z_{2,t}$: Stage2 inventory level after regular production during period t.
$\overline{X}_{2,t}$: The stage2 inventory level at the start of overtime, after receiving Demand from stage1 during period t.
$\overline{Y}_{2,t}$: The stage2 overtime production quantity during period t.
$\overline{Z}_{2,t}$: The stage2 inventory level after overtime production during period t
$X_{2,t}^1$: The stage2 inventory position at the start of outsourcing during period t.
$Y_{2,t}^1$: The stage2 outsourcing quantity during period t.
$Z_{2,t}^1$: The stage2 inventory level after outsourcing during period t.

Every production process entails various cost implications. At stage1 set up costs are assessed for production ((C_1) , holding cost (h_1) and since backordering is allowed at stage1, we've backordering costs (b_1). At stage2 backordering is not allowed, however, linear costs are assessed for production (c_2) and holding cost (h_2), overtime production incurs linear cost (c_0) plus fixed costs (k_0) and outsourcing incurs linear costs (c_3) plus fixed cost (k_3). All the costs are assumed to be discounted every period by a factor α, with $0 < \alpha < 1$. We now present our sequence of activities at both stages as below:

7. DECENTRALIZED MODEL

The two stages of supply chain management are independent firms and each seeking to minimize its own costs. These costs includes: linear cost of production. Holding cost and backordering costs respectively. We determine the optimal inventory control policy for stage1 and proceed to show that it is a base-stock policy. We establish that the optimal inventory control policy for stage2 is a base-stock policy under the assumptions below: -That overtime production is the only method of expediting available to stage2.

7.1 Stage1 Optimal Policy Under Decentralized Control

Under decentralized control, stage1 is an independent firm and makes decisions based on the initial inventory available, X_1. He incurs linear cost of production, c_1, holding cost, h_1 and backordering cost b_1. All the variables discussed in this section occur during the same period, t so we drop the subscript t for notational convenience.

The 1-period costs experienced by stage1 are:

$$g_{1,dec.}(X_1, Z_1, D) = C_1(Z_1 - X_1) + h_1(Z_1 - D)^+ + b_1(Z_1 - D)^-$$

$$= C_1Z_1 - C_1X_1 + h_1(Z_1 - D)^+ + b_1(Z_1 - D)^-.$$

With $Z_1 \geq X_1$, clearly, $g_{1,dec.}(\cdot) \geq 0$. Hence the optimal cost function $J^*_{1,dec.}(X_1)$ satisfies

$$J^*_{1,dec.}(X_1) = \min_{Z_1 \geq X_1} \lim_{N\to\infty} E_D\{\sum_{k=0}^{N-1} \alpha^k g_{1,dec.}(X_1, Z_1, D)\}.$$

The argument that minimizes this equation is the optimal inventory control policy we seek. To determine this policy, we use a technique similar to Veinott(1965), Huggins(2002). We proceed as follows:

Move the $-C_1X_1$ term back to the previous period as $-\alpha C_1(Z_1 - D)$ and determine our moved one period costs as:

$$\begin{aligned} g_{1,dec,v}(X_1, Z_1, D) &= C_1Z_1 - \alpha C_1(Z_1 - D) + h_1(Z_1 - D)^+ + b_1(Z_1 - D)^- \\ &= C_1Z_1 - \alpha C_1Z_1 + \alpha C_1D + h_1(Z_1 - D)^+ + b_1(Z_1 - D)^- \\ &= (1 - \alpha)C_1Z_1 + \alpha C_1D + h_1(Z_1 - D)^+ + b_1(Z_1 - D)^- \\ & \quad Z_1 \geq X_1. \end{aligned}$$

We use this method to derive the optimal policy for stage1 step-by-step for two reasons:

i) The more complicated derivations later in the thesis tend to follow the same steps and we feel that the proof of lemma1 below is a good introduction to this methodology.

ii) One of the steps we will frequently use is to "move" a term backward to the previous period.

Lemma. The optimal policy that solves $J^*_{1,dec.}(X_{1,0})$ also solves $J^*_{1,dec.,v}(X_{1,0})$. And $J^*_{1,dec.}(X_{1,0}) = -C_1X_{1,0} + J_{1,dec.,v}(X_{1,0})$.

Proof:

$$\begin{aligned}
J^*_{1,dec}(X_{1,0}) &= \min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-1}\alpha^k g_{1,dec}(X_{1,k},Z_{1,k},D)\}\\
&= \min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\alpha^k[C_1(Z_{1,k}-X_{1,k})+h_1(Z_{1,k}-D)^+ + b_1(Z_{1,k}-D)^-]\}\\
&= \min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\alpha^k C_1(Z_{1,k}-\alpha^k C_1 X_{1,k})+\alpha^k[h_1(Z_{1,k}-D)^+ + b_1(Z_{1,k}-D)^-]\}\\
&= \min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-1}\alpha^k(-C_1X_{1,k})\}\\
&\quad +\min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-1}\alpha^k[C_1Z_{1,k}+h_1(Z_{1,k}-D)^+ + b_1(Z_{1,k}-D)^-]\}
\end{aligned}$$

Thus,

$$\begin{aligned}
J^*_{1,dec}(X_{1,0}) &= -C_1X_{1,0}+\min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-2}\alpha^k[C_1Z_{1,k}+h_1(Z_{1,k}-D)^+ + b_1(Z_{1,k}-D)^-\\
&\quad -\alpha C_1(Z_1-D)]+\alpha^{N-1}[C_1Z_{1,N-1}+h_1(Z_{1,N-1}-D_{1,N-1})^+\\
&\quad +b_1(Z_{1,N-1}-D_{1,N-1})^-\}\\
&= -C_1X_{1,0}+\min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-2}\alpha^k[g_{1,dec.,v}(X_{1,k},Z_{1,k},D)^-]\}\\
&\quad +\min_{\pi\in\Pi}\lim_{N\to\infty}\{\alpha^{N-1}[C_1Z_{1,N-1}+h_1(Z_{1,N-1}-D_{1,N-1})^+ + b_1(Z_{1,N-1}-D_{1,N-1})^-]\}\\
&= -C_1X_{1,0}+\min_{\pi\in\Pi}\lim_{N\to\infty} E_D\{\sum_{k=0}^{N-2}\alpha^k[g_{1,dec,v}(X_{1,k},Z_{1,k},D)^-]\}+0\\
&= -C_1X_{1,0}+J_{1,dec,v}(X_{1,0}).
\end{aligned}$$

We now consider the optimal cost function $J^*_{1,dec,v}(X_1,Z_1,D)$.

$$\begin{aligned}
J^*_{1,dec,v}(X_1,Z_1,D) &= \min_{Z_1\geq X_1} E_D\{g_{1,dec,v}(X_1,Z_1,D)+\alpha J^*_{1,dec,v}(Z_1-D)\}\\
&= \min_{Z_1\geq X_1}\{E_D[g_{1,dec,v}(X_1,Z_1,D)]+\alpha E_D[J^*_{1,dec,v}(Z_1-D)]\}\\
&= \min_{Z_1\geq X_1}\{G_{1,dec,v}(Z_1)+\alpha E_D[J^*_{1,dec,v}(Z_1-D)]\}.
\end{aligned}$$

From

$$g_{1,dec,v}(X_1,Z_1,D)=(1-\alpha)C_1Z_1+\alpha C_1D+h_1(Z_1-D)^+ + b_1(Z_1-D)^-,$$

it is clear that $G_{1,dec,v}(Z_1)$ is convex and so $-G_{1,dec,v}(Z_1)$ unimodal. We now apply the method used by Zheng (1991) to show that the optimal inventory control policy at stage1 is a base - stock policy. To apply the results from Zheng's paper, we need that $-G_{1,dec,v}(Z_1)$ is unimodal and that $G_{1,dec,v}(Z_1)\to\infty$ as $|Z_1|\to\infty$. For $Z_1<0$, the slope(in the discrete sense) of $G_{1,dec.,v}(Z_1)$ is $-C_3+(1-\alpha)C_1<0$ by assumption (A6). Thus, as $Z_1\to-\infty$, $G_{1,dec,v}(Z_1)\to\infty$ as $Z_1\to\infty$, the slope of $G_{1,dec,v}(Z_1)$ becomes $(1-\alpha)c_1+h_1>0$ and thus $G_{1,dec,v}(Z_1)\to\infty$, hence we have that the optimal inventory control policy at stage1 is a base-stock policy. We now define the base-stock level as $S^*_{1,dec}$. We assume that the initial inventory is not more than this value that is, $X_1 \leq S^*_{1,dec}$, we can calculate $J^*_{1,dec}(X_1)$.

$$\begin{aligned}
J^*_{1,dec}(X_1) &= -C_1X_1 + J^*_{1,dec,v}(X_1) \\
&= -C_1X_1 + \min_{Z_1 \geq X_1}\{G_{1,dec,v}(Z_1) + \alpha E_D[J^*_{1,dec,v}(Z_1 - D)]\} \\
&= -C_1X_1 + \min_{Z_1 \geq X_1}\{G_{1,dec,v}(S^*_{1,dec}) + \alpha E_D[J^*_{1,dec,v}(S^*_{1,dec} - D)]\} \\
&= -C_1X_1 + G_{1,dec,v}(S^*_{1,dec}) + \alpha E_D[J^*_{1,dec,v}(S^*_{1,dec} - D)] \\
&= -C_1X_1 + G_{1,dec,v}(S^*_{1,dec}) + \alpha G_{1,dec,v}(S^*_{1,dec}) + \alpha^2 E_D[J^*_{1,dec,v}(S^*_{1,dec,v} - D)] \\
&= -C_1X_1 + G_{1,dec,v}(S^*_{1,dec}) + \alpha G_{1,dec,v}(S^*_{1,dec})[1 + \alpha + \alpha^2 + \cdots] \\
&= -C_1X_1 + \frac{G_{1,dec,v}(S^*_{1,dec})}{1-\alpha}.
\end{aligned}$$

Hence under decentralized control, the optimal policy at stage1 is to order-up to $S^*_{1,dec}$ every period. Note that due to the base-stock policy, stage1 will pass the exact demand it experiences back to stage2, and stage2 will face the same demand that stage1 faces.

7.2 Stage2 Optimal Policy Under Decentralized Control

Under the decentralized control, stage2 is an independent firm. The manager makes decisions based on the initial inventory available, X_2 and the potential costs incurred. Stage2 faces the same demand distribution as stage1. We assume that overtime production is the only method of filling shortages. Thus, the overtime decision is straightforward. Note that per unit cost of overtime production is more than that of regular production, hence it will never be cost-effective to produce more than the shortage with overtime production.

During period t, stage2 will receive order $Y_{1,t+1} = D_t$ from stage1. At the beginning of overtime, the inventory level is, $\overline{X}_{2,t} = Z_{2,t} - Y_{1,t+1}$ if $\overline{X}_{2,t} < 0$, then overtime production must be employed. This quantity is $(Z_{2,t} - D_t)^-$. All the variables occur during the same period, t, so we drop the subscript. The one period costs experienced by stage2 are

$$g_{2,dec}(X_2, Z_2, D) = C_2(Z_2 - X_2^+) + h_2(Z_2 - D)^+ + C_0(Z_2 - D)^- + K_0\delta(Z_2 - D)^-$$

with $Z_2 \geq X_2^+$. Clearly, $g_{2,dec}(\cdot) \geq 0$ and hence the optimal cost function $J^*_{2,dec}(X_2)$ satisfies

$$J^*_{2,dec}(X_2, Z_2, D) = \min_{Z_2 \geq X_2} E_D\{g_{2,dec}(X_2, Z_2, D) + \alpha J^*_{2,dec.}(Z_2 - D)\}.$$

To determine the optimal policy, we move the $-C_2X_2$ term back to the previous period as $-\alpha C_2(Z_2 - D)^+$ as we did in the previous section. We now define our moved one period costs as,

$$g_{2,dec.,v}(X_2, Z_2, D) = C_2Z_2 + K_0\delta[(Z_2 - D)^-] + C_0(Z_2 - D)^- + (h_2 - \alpha C_0)(Z_2 - D)^+$$

with $Z_2 \geq X_2^+$. Note that the optimal policy that solves $J^*_{2,dec}(X_2)$ also solves $J^*_{2,dec.,v}(X_2)$.
And $J^*_{2,dec.}(X_2) = -C_2X_{2,0} + J_{2,dec.,v}$

$$J^*_{2,dec.,v}(X_2) = \min_{Z_2 \geq X_2} E_D\{g_{2,dec.,v}(X_2, Z_2, D) + \alpha J^*_{2,dec.,v}(Z_2 - D)\}$$

$$= \min_{Z_2 \geq X_2}\{G_{2,dec.,v}(Z_2) + \alpha E_D[J^*_{2,dec.,v}(Z_2 - D)]\}$$

where $G_{2,dec.,v}(Z_2) = E_D\{g_{2,dec.,v}(X_2, Z_2, D)\}$.
Again we apply the result from Zheng as before, we need to show that $G_{2,dec.,v}(Z_2) \to \infty$ as $|Z_2| \to \infty$ and

that $-G_{2,dec.,v}(Z_2)$ is unimodal or that $G_{2,dec.,v}(Z_2)$ is quasiconvex. For $Z_2 < 0$, the slope of $G_{2,dec.,v}(Z_2)$ is $C_2 - C_0 < 0$ by A3, thus, as $Z_2 \to -\infty$, $G_{2,dec.,v}(Z_2) \to \infty$. As $Z_2 \to +\infty$, the slope of $G_{2,dec.,v}(Z_2) \to \infty$ becomes $h_2 + (1-\alpha)C_2 > 0$, and thus $G_{2,dec.,v}(Z_2) \to \infty$. Hence we have from Zheng that the optimal inventory control policy at stage2 is a base-stock policy.

We now define the optimal base-stock level as $S^*_{2,dec}$, under the assumption that the initial inventory is not more than this value, $X_2 \leq S^*_{2,dec}$. We compute $J^*_{2,dec.}(X_2)$,

$$\begin{aligned}
J^*_{2,dec.}(X_2) &= -C_2X_2^+ + J_{2,dec.,v}(X_2) \\
&= -C_2X_2^+ + \min_{Z_2 \geq X_2}\{G_{2,dec,v}(Z_2) + \alpha E_D[J^*_{2,dec,v}(Z_2 - D)]\} \\
&= -C_2X_2^+ + G_{2,dec,v}(S^*_{2,dec}) + \alpha E_D[J^*_{2,dec,v}(S^*_2 - D)] \\
&= -C_2X_2^+ + G_{2,dec,v}(S^*_{2,dec}) + \alpha G_{2,dec,v}(S^*_{2,dec}) + \alpha^2 E_D[J^*_{2,dec,v}(S^*_{2,dec} - D)] \\
&= -C_2X_2^+ + G_{2,dec,v}(S^*_{2,dec})[1 + \alpha + \alpha^2 + \cdots] \\
&= -C_2X_2^+ + \frac{G_{2,dec,v}(S^*_{2,dec})}{1-\alpha}.
\end{aligned}$$

Hence under decentralized control with overtime as the only expediting option, the optimal policy at stage2 is to order-up to $S^*_{2,dec}$ every period.

8. CONCLUSION

Note that the decisions are made separately by the two stages, as in our decentralized model. Under the assumptions of the centralized model, all decisions about $Z_{1,t+1}$, $Z_{2,t}$ and $Z_{2,t+1}$ are made at the same time, where stage 1 makes its decision in the time line. In our model, we make the following assumptions. First, as mentioned above, we assume a discount factor α, with $0 < \alpha < 1$. Second, we assume that demand is discrete, non-negative, stationary, and from a discrete probability distribution. We assume that the expected value of demand, μ, is positive and finite. Third, we assume that per unit cost of overtime production at stage 2 is greater than per unit cost of regular production at stage 2. Fourth, we assume that the cost of backordering at stage 1 is not so small that it is cheaper to always backorder than to produce. All of these assumptions are fairly standard.

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