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S-(H,Ω) CONJUGATE DUALITY THEORY IN MULTIOBJECTIVE NONLINEAR OPTIMIZATION[1]

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Abstract: Based on strong efficiency instead of weak efficiency or efficiency, this paper gives the definitions and some fundamental properties of the strong supremum and infimum sets. The concepts, some properties and their relationships of s-(H,Ω) conjugate maps, s-(H,Ω)-subgradients, s-Γ_H^p(Ω)-regularitions of vector-valued point-to-set maps are provided, and a new duality theory in multiobjective nonlinear optimization------s-(H,Ω) Conjugate Duality Theory is established by means of the s-(H,Ω) conjugate maps. The concepts and their relationships between the strong efficient solutions of the primal and dual problems and the strong saddle-points of the s-(H,Ω)-Lagrangian map are developed. Finally, some possible further research works are given.

Key words: conjugate duality theory, multiobjective optimization, strong efficiency

1. INTRODUCTION

In the duality theory of nonlinear programming problems, conjugate functions play important roles [1,2,8]. So, in order to discuss duality in multiobjective optimization problems, we need to introduce an extended notion of "conjugate" which fits well the multiobjective problems. Tanino and Sawaragi [10] have developed a duality theory in multiobjective convex programming problems under the Pareto optimality criterion. However, their duality theory is not reflexive in the sense analogous to [1 , 2]. The nonreflexivity of their duality seems to be caused by the fact that the objective functions in the primal problems are vector-valued, while those in the dual problems are set-valued. If we are concerned with the reflexivity of the duality theory, we will have to start from set-valued objective functions in the primal problems. Then a new problem comes about: what kinds of set-valued functions are suitable for objective functions ? Feng [3] extended the results of [10] to a more general framework, and provided a reflexive duality theory in multiobjective optimization based on efficiency. Based on weak efficiency rather than efficiency, Kawasaki [6,7] developed some interesting results by defining conjugate and subgradients via weak supremum. In this article, a new duality theory for weak efficiency is developed with the help of the weak supremum and the generalization of the conjugate relations discussed in [6]. Feng[3] extended Kawasaki’s work to a more general case.

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In section 2, the concepts of the strong infimum and strong supremum are developed. In section 3, we shall give the definitions of s-(H,Ω) conjugate maps and s-(H,Ω)-subgradients and s-Γ_H^p(Ω)-regulations of the point-to-set map, with the discussion of their fundamental properties. The strong saddle-point problem is also discussed in this section. In section 4, the s-(H,Ω) conjugate duality theory is developed which states the relationships between the primal and dual problems. In section 5, the s-(H,Ω)-Lagrangian map is defined and its strong saddle-point problem is discussed. In section 6, some possible extensions are suggested.

Before going further, we introduce the following notations. Let R^p be the p-dimensional Euclidean space, and ε and -ε be the p-dimensional points consisting of +∞ and -∞, respectively, and $\underline{R}^p=R^p\cup\{\varepsilon,-\varepsilon\}$, and R_+^p ($\underline{R}_+^p$) be the subset of R^p ($\underline{R}^p$) with the elements being nonnegative, $R_-^p=-R_+^p$, $\underline{R}_-^p=-\underline{R}_+^p$. For any y^1 and y^2 in $\underline{R}^p$, we denote $y^1\leq=y^2$ if $y^1=y^2$ or $y^1=-\varepsilon$ or $y^2=\varepsilon$ or $y^2-y^1\in R_+^p\setminus\{0\}$; we denote $y^1\leq y^2$ if $y^1\leq=y^2$ and $y^1\neq y^2$, and $y^1<y^2$ if each component of y^2-y^1 is positive. For any $a\in\underline{R}^p$, $B\subseteq R^p$ and $C\subseteq R^p$, we define: $a+\varepsilon=+\varepsilon$; $a+B=\{a+b\in R^p : b\in B\}$; $B+C=\{b+c\in\underline{R}^p : b\in B,c\in C\}$.The p-dimensional point consisting of a common element $a\in R$ is simply denoted by **a** in this paper. Suppose $X\subseteq R^p$ and $F:R^n\to R^p$, sometimes, we denote F(X) or $\cup_X F(x)$ to be $\cup_{x\in X}F(x)$. Additionally, considering the length of the paper, we omit the proofs of all the theorems.

2. STRONG SUPREMUM AND INFIMUM SETS AND THEIR PROPERTIES

Definition 2.1 Let $Y\subseteq\underline{R}^p$. $y'\in\underline{R}^p$ is said to be a strong supremum point of Y if (1) $y'\in\underline{R}^p$; (2) $y\leq=y'$ for $\forall y\in Y$; (3) $y'\leq=y''$ as long as $y''\in\underline{R}^p$ and satisfies $y\leq=y''$ for $\forall y\in Y$. The set of strong supremum points of Y is denoted by s-SupY , which will be called the strong supremum set of Y. Similarly, $y'\in\underline{R}^p$ is said to be a strong infimum point of Y if (1) $y'\in\underline{R}^p$; (2) $y'\leq=y$ for $\forall y\in Y$; (3) $y''\leq=y'$ as long as $y''\in\underline{R}^p$ and satisfies $y''\leq=y$ for $\forall y\in Y$. The set of strong infimum points of Y is denoted by s-InfY , which will be called the strong infimum set of Y.

Theorem 2.1 (Existence and Uniqueness) For $\forall Y\subseteq\underline{R}^p$, its supremum point and infimum point exist and are unique, that is, the sets s-SupY and s-InfY are all single element set. Therefore, we will simply denote the sole element by s-SupY and s-InfY, respectively. Furthermore, if s-SupY∈Y, we will denote s-SupY by s-MaxY. Similarly, If s-InfY∈Y, we will denote s-InfY by s-MinY.

It is easily known that for any $Y\subseteq\underline{R}^p$, s-SupY=-s-Inf(-Y), s-InfY=-s-Sup(-Y), furthermore, if Y=∅, then s-SupY=-ε, s-InfY=+ε; if +ε∈Y, then s-SupY=+ε; if -ε∈Y, then s-SupY=-ε.

Theorem 2.2 Let $Y\in\underline{R}^p$. Then

(1) For each $y\in Y\setminus\{-\varepsilon\}$, there exists a point $t\in\underline{R}_+^p$ such that y+t=s-SupY. In particular, for each $y\in Y\cap R^p$, the corresponding t is unique .

(2) For each $y\in Y\setminus\{+\varepsilon\}$, there exists a point $t\in\underline{R}_-^p$ such that y+t=s-InfY. In particular, for each $y\in Y\cap R^p$, the corresponding t is unique .

Theorem 2.3 For any Y_1 and $Y_2\subseteq\underline{R}^p$, if $y_1\leq=y_2$ for $\forall y_1\in Y_1$ and $y_2\in Y_2$, then s-Sup$Y_1\leq=$s-InfY_2. If s-Inf$Y_2\in Y_1$ or s-Sup$Y_1\in Y_2$, then s-SupY_1=s-InfY_2. Particularly, if Y_1 and $Y_2\subseteq R^p$, and $Y_1\cap Y_2\neq\emptyset$, then s-SupY_1 and s-InfY_2 are reachable, that is , s-SupY_1=s-MaxY_1, and s-InfY_2=s-MinY_2.

Theorem 2.4 If $Y_1,Y_2 \subseteq \underline{R}^p$ and $Y_1 \subseteq Y_2$, then s-Sup$Y_1 \leq$=s-SupY_2.

3. S-(H,Ω) CONJUGATE MAPS, S-(H,Ω)-SUBGRADIENTS AND S-(H,Ω)-REGULARITIONS

Throughout this paper, we shall designate by X and Y two locally convex Hausdorff spaces (finite-dimensional). Usually, we can suppose that $X=R^n$ and $Y=R^m$.

Definition 3.1 Let $F:X \to \underline{R}^p$, Ω be a family of vector-valued functions ω:X→Y, and H be a family of vector-valued functions $h:Y \to \underline{R}^p$, closed s-Sup pointwise, i.e., if H'⊆H, then s-Sup{H'}∈H, where s-Sup is taken pointwisely, that is, for ∀y∈Y, (s-Sup{H'})(y)=s-Sup{H'(y)}. Under the above assumption, the s-(H,Ω) conjugate map s-F^c :Ω→H of F is defined by the formula:

$$\text{s-}F^c(\omega)=\text{s-Sup}\{h:h\in H, h\otimes\omega\leq=F\} \quad \text{for } \forall\omega\in\Omega$$

that is, s-F^c (ω)(y)=s-Sup{h(y): h∈H, h⊗ω≤=F} for ∀ω∈Ω and y∈Y

Inversely, let G:Ω→H, the s-(H,X) conjugate map s-G* :$X\to\underline{R}^p$ of G is defined by the formula:

$$\text{s-}G^*(x)=\text{s-Sup}\{h(\omega(x)): h\in H, \omega\in\Omega, h\leq=G(\omega)\} \quad \text{for } \forall x\in X$$

It is easily verified that s-G*(x)=s-Sup $\cup_\omega$ G(ω)(ω(x)) for ∀x∈X

Moreover, the s-(H,X) conjugate map of s-F^c is called the s-(H,Ω) biconjugate map of F, and is denoted by s-F^{c*} , i.e., s-F^{c*} :$X\to\underline{R}^n$, and s-F^{c*} =s-(s-F^c)* , and s-F^{c*}(x)=s-Sup $\cup_{\omega\in\Omega}$ s-F (ω)(ω(x)) for ∀x∈X.

Definition 3.2 (s-ΓpH (Ω) Family) The set of the point-to-point maps $F:X\to\underline{R}^n$ which can be written in the following form:

$$F(x)=\text{s-Sup}\{h(\omega(x)): h\in H', \omega\in\Omega'\} \quad \text{for } \forall x\in X$$

is denoted by s-$\Gamma^p{}_H$ (Ω), where H'⊂H, Ω'⊆Ω and Ω' is nonempty.

Definition 3.3 (s-ΓpH (Ω)-normalization) Let $F:X\to\underline{R}^n$. The point-to-point map s-$\tilde{F}$ (H,Ω):$X\to\underline{R}^n$ defined by the following formula is called the s-$\Gamma^p{}_H$ (Ω)-normalization of F:

$$\text{s-}\tilde{F}(H,\Omega)(x)=\text{s-Sup}\{h(\omega(x)): h\in H, \omega\in\Omega, h\otimes\omega\leq=F\} \quad \text{for } \forall x\in X$$

Let H_L ={h_b :h_b (t)=t+b (∀t∈R^p), b∈$\underline{R}^p$ }, Ω ={ω* : ω*(x)=<<x,x*>> (∀x∈X), x*∈X*}where X* is the dual space of X, and <•,•> is a bilinear pairing between X and X* , and <<•,•>>=(<•,•>,<•,•>,...,<•,•>).

Theorem 3.1 Let $H=H_L$, $Y=R^p$, $F:X\to\underline{R}^p$, and G:Ω→H. Then s-F^c can be written : s-F^c (ω)(t)=t-s-Sup{ $\cup_x${ω(x)-F(x)}} for ∀ω∈Ω and t∈Y. Similarly, s-G* can be written: s-G*(x)=s-Sup $\cup_\omega$ {ω(x)+$\underline{A}$(ω)} for ∀x∈X, where $\underline{A}$(ω)=s-InfA(ω), and A(ω) satisfies G(ω)(t)=t+A(ω) for ∀t∈Y and

$\omega\in\Omega$. In particular, s-F^{c}* can be written: s-F^{c}*(x)=s-Sup $\cup_{\omega}$ {ω(x)- $\overline{F}$ (ω)} for $\forall x\in X$, where $\overline{F}$ (ω)=s-Sup $\cup_{\omega}$ {ω(x)-F(x)} for $\forall\omega\in\Omega$.

The relationships between the s-(H,Ω) conjugate map and s-$\Gamma^{P}{}_{H}$ (Ω)-normalization of F are given as follows.

Theorem 3.2 Let F:X→$\underline{R}^{p}$, and G:Ω→H. Then

(1) s-$\tilde{\tilde{F}}$ (H,Ω)∈s-$\Gamma^{P}{}_{H}$ (Ω) and s-G* ∈s-$\Gamma^{P}{}_{H}$ (Ω).

(2) s-$\tilde{\tilde{F}}$ (H,Ω)≤=F

(3) s-$\tilde{\tilde{F}}$ (H,Ω)=s-F^{c}*

(4) If F'∈s-$\Gamma^{P}{}_{H}$ (Ω) and F'≤=F, then F' ≤=s- $\tilde{\tilde{F}}$ (H,Ω)

(5) s-$\tilde{\tilde{F}}$ (H,Ω)=F if and only if F∈s-$\Gamma^{P}{}_{H}$ (Ω).

(6) s-F^{c} (ω)(ω(x))≤=F(x) for $\forall\omega\in\Omega$ and $x\in X$.

Similar discussions can be made for G:Ω→P(H).

Definition 3.3 (s-ΣpH (Ω) Family) The set of the point-to-point maps G:Ω→H which can be written: G(ω)=s-Sup{h: h∈H_{ω} } for $\forall\omega\in\Omega$, where $H_{\omega}\subseteq H$ for each $\omega\in\Omega$ is denoted by s-$\Sigma^{P}{}_{H}$ (Ω)

Definition 3.4 (s-ΣpH (Ω)-normalization) Let G:Ω→P(H). The point-to-point map s-$\tilde{\tilde{G}}$ (H,Ω):Ω→H defined by:

s-$\tilde{\tilde{G}}$ (H,Ω)(ω)=s-Sup{h: h∈H,h≤=G(ω)}=s-Sup{h: h∈H, h(t)≤=G(ω)(t) $\forall t\in Y$} for $\forall\omega\in\Omega$

is called the s-$\Sigma^{P}{}_{H}$ (Ω)-normalization of G.

Dually to theorem 3.2, we have:

Theorem 3.5 Let G:Ω→H, F:X→$\underline{R}^{p}$. Then

(1) s-$\tilde{\tilde{G}}$ (H,Ω)∈s-$\Sigma^{P}{}_{H}$ (Ω) and s-F^{c}∈s-$\Sigma^{P}{}_{H}$(Ω)

(2) s-$\tilde{\tilde{G}}$ ≤= s-G*^{c}

(3) s-$\tilde{\tilde{G}}$ * =s-G*

(4) s-$\tilde{\tilde{G}}$ (H,Ω)≤=G

(5) if G'∈s-$\Sigma^{P}{}_{H}$ (Ω) and G'≤=G, then G′≤= s-$\tilde{\tilde{G}}$ (H,Ω)

(6) s-$\tilde{\tilde{G}}$ (H,Ω)=G if and only if G∈s-$\Sigma^{P}{}_{H}$ (Ω)

(7) if s-SupG(ω)=G(ω) for some $\omega\in\Omega$, then G(ω)(ω(x))≤=s=-G*(x) for $\forall x\in X$.

Definition 3.5 F:X→$\underline{R}^{p}$ is said to be s-(H,Ω)-subdifferentiable if there exist h∈H and ω∈Ω such that h(ω(x))=y and h⊗ω≤=F. And such ω∈Ω is called the s-(H,Ω)-subgradient of F. The set of all the s-(H,Ω)-subgradients of F is called the s-(H,Ω)-subdifferential of F , which is denoted by s-∂_{Ω} F(x).

Definition 3.6 $G:\Omega\to H$ is said to be s-(H,X)-subdifferentiable, if there exists $x\in X$ such that $h(\omega(x))\in$s-$G^*(x)$, and $h\otimes\omega\leq=$s-G^* . And such $x\in X$ is called the s-(H,X)-subgradient of G . The set of all the s-(H,X)-subgradients of G is called the s-(H,X)-subdifferential of G, and will be denoted by s-$\partial_X G(\omega)$.

Theorem 3.6 Let $F:X\to \underline{R}^p$.

(1) $\omega\in$s-$\partial_\Omega F(x)$ if and only if $F(x)=$s-$F^c(\omega)(\omega(x))$.

(2) s-$\partial_\Omega F(x)\subseteq$s-$\partial_\Omega$ s-$F^{c}*(x)$ for $\forall x\in X$ and $y\in \underline{R}^P$.

(3) If s-$F^{c}*(x)=F(x)$, s-$F^c=$s-F^{c*c} , then s-$\partial_\Omega F(x)=$s-∂_Ω s-$F^{c}*(x)$ for $\forall y\in$s-$F^{c}*(x)$.

(4) Suppose that Ω contains the element 0_Ω satisfying $0_\Omega(x)=0\in Y$ for $\forall x\in X$, and H has the following property: for each $a\in\underline{R}^p$, there exists $h_a\in H$ such that $h_a(0)=a$. Then $0_\Omega\in$s-$\partial_\Omega F(x)$ if and only if $F(x)=$s-Min $\cup_x F(x)$.

Theorem 3.7 Let $G:\Omega\to P(H)$.

(1) If $x\in$s-$\partial_X G(\omega)$, then $h(\omega(x))=G(\omega)(\omega(x))=$s-$G^{*c}(\omega)(\omega(x))$.

(2) If s-$\partial_X G(\omega)\neq\emptyset$, then there exists $h'\in H$ such that s-∂_X s-$G^{*c}(\omega)\neq\emptyset$.

(3) If s-$G^{*c}(\omega)=G(\omega)$, then s-$\partial_X G(\omega)=$s-∂ s-$G^{*c}(\omega)$.

The relationship between the s-(H,Ω)-subgradient and s-(H,X)-subgradient is as follows.

Theorem 3.8 Let $F:X\to P(\underline{R}^p)$, $G:\Omega\to P(H)$.

(1) If $\omega\in$s-$\partial_\Omega F(x)$, then $x\in$s-∂_Xs-$F^c(\omega)$. Inversely, if $F\in$s-$\Gamma^p{}_H(\Omega)$ and $x\in$s-∂_X s-$F^c(\omega)$, then $\omega\in$s-$\partial_\Omega F(x)$.

(2) If $x\in$s-$\partial_X G(\omega)$, then there $\omega\in$s-∂_Ω s-$G^*(x)$. Inversely, if s-$G^{*c}(\omega)=G(\omega)$, $\omega\in$s-∂_Ωs-$G^*(x)$, then $x\in$s-$\partial_X G(\omega)$.

In the following, we will discuss the strong saddle-point problem of point-to-point maps. Let point-to-point map L be $L:X\times Y\to\underline{R}^p$, where X and Y are two finite-dimensional locally convex Hausdorff spaces.

Definition 3.9

The point $(x',y')\in X\times Y$ is called the strong saddle-point of map L if $L(x',y)\leq=L(x',y')\leq=L(x,y')$ for $\forall x\in X$ And $y\in Y$.

When p=1 and $L:X\times Y\to R$, the above definition coincides with that of the saddle-point, in the sense, of a function.

In the following, for convenience, we shall use the notations given by:

s-$SupL_x$ =s-Sup $\cup_y L(x,y)$; s-$InfL_y$ =s-Inf $\cup_x L(x,y)$;

s-Infs-SupL=s-Inf $\cup_x$ s-$SupL_x$; s-Sups-InfL=s-Sup $\cup_y$ s-$InfL_y$;

s-Mins-SupL=s-Min $\cup_x$ s-$SupL_x$; s-Maxs-InfL=s-Max $\cup_y$ s-$InfL_y$

By the definition of the strong saddle-point, (x,y) is the strong saddle-point of L if and only if $L(x,y)=$s-Max $\cup_y L(x,y)$ = s-Min $\cup_x L(x,y)$ if and only if $L(x,y)=$s-Max $\cup_y$ s-Min $\cup_x L(x,y)$ =s-Min $\cup_x$

s-Max $\cup_y$ L(x,y).

4. S-(H,Ω) DUALITY IN MULTIOBJECTIVE OPTIMIZATION

Let X,U, and Y be three locally convex Hausdorff spaces, where X is called the decision space and U is called the perturbation space. Let Φ be a map from X×U into $\underline{R}^p$, i.e., $\Phi: X\times U\rightarrow \underline{R}^p$. We assume that Φ is not identically empty on X×{0}.

Let Ω_X and Ω_U be two families of functions from X into Y and U into Y, respectively. Especially, we denote $\Omega'=\Omega_X \otimes\Omega_U$ and $\Omega=\Omega_U$, where Ω' is a family of functions from X×U into Y and $\omega'\in\Omega'$ if and only if there exist $\omega\in\Omega_X$ and $\theta\in\Omega_U$ such that $\omega'=\omega\oplus\theta$, that is, $\omega'(x,u)=\omega(x)+\theta(u)$ for $\forall(x,u)\in X\times U$. For $\omega'=\omega\oplus\theta$, we sometimes write it $\omega'=(\omega,\theta)$. In particular, we assume that Ω_X contains the element 0_X satisfying $0_X(x)=0_Y\in Y$, where 0_Y is the zero element of Y. Finally, Let H be a family of functions from Y into $\underline{R}^p$, closed s-Sup pointwise.

Then we consider the following optimization problems with set-valued objective functions:

(MO) The original problem: Find $x\in X$ such that $\Phi(x,0)=$s-Min $\cup_x \Phi(x,0)$

(MP) The primal problem: Find $x\in X$ such that $\Phi(x,0)=$s-Inf $\cup_x \Phi(x,0)$

(MD') The s-(H,Ω) conjugate dual problem of (MO): Find $\theta\in\Omega$ such that

s-$\Phi^c(0_X,\theta)(\theta(0))$=s-Max $\cup_\theta$ s-$\Phi^c(0_X,\theta)(\theta(0))$

(MD) The s-(H,Ω) conjugate dual problem of (MP): Find $\theta\in\Omega$ such that

s-$\Phi^c(0_X,\theta)(\theta(0))$=s-Sup $\cup_\theta$ s-$\Phi^c(0_X,\theta)(\theta(0))$

We call solutions of (MO), (MP), (MD'), and (MD) in X or Ω strong efficient solutions, respectively, which will be denoted by s-Eff(MO), s-Eff(MP), s-Eff(MD') and s-Eff(MD), respectively. Moreover, denote

s-Min(MP)=s-Min $\cup_x \Phi(x,0)$

s-Inf(MP)=s-Inf $\cup_x \Phi(x,0)$

s-Max(MD)=s-Max $\cup_\theta$ s-$\Phi^c(0_X,\theta)(\theta(0))$

s-Sup(MD)=s-Sup $\cup_\theta$s-$\Phi^c(0_X,\theta)(\theta(0))$

Theorem 4.1 s-Eff(MO)=s-Eff(MP) ; s-Eff(MD')=s-Eff(MD).

By virtue of this theorem, we have only to consider the duality between (MP) and (MD) instead of (MO) and (MD'). The first duality is the following weak duality.

Theorem 4.2 (s-(H,Ω) Weak Duality Theorem)

(1) s-Max(MD)≤=s-Min(MP)

s-Max(MD)$\leq$=Φ(x,0) for $\forall x\in X$

s-$\Phi^c(0_X,\theta)(\theta(0))\leq$=s-Min(MP) for $\forall\theta\in\Omega$;

s-Max(MD)$\leq$=$\cup_x \Phi(x,0)$; $\cup_\theta$ s-$\Phi^c(0_X,\theta)(\theta(0))\leq$=s-Min(MP) .

(2) If $x\in X$ and $\theta\in\Omega$ satisfy s-$\Phi^c(0_X,\theta)(\theta(0))=\Phi(x,0)$, then $x\in$s-Eff(MP) and $\theta\in$s-Eff(MD).

(3) If $x\in X$ satisfies $\Phi(x,0)$=s-Max(MD), then $x\in$s-Eff(MP) and s-Min(MP)=s-Max(MD). Substituting s-Max for s-Sup, the conclusion still holds.

(4) If $\theta\in\Omega$ satisfies s-$\Phi^c(0_X,\theta)(\theta(0))$=s-Min(MP), then $\theta\in$s-Eff(MD) and s-Min(MP)=s-Max(MD). Substituting s-Min for s-Inf, the conclusion still holds.

Theorem 4.3 s-Sup(MD)$\leq$= s-Inf(MP).

The perturbed problem relative to (MP) and (MD) is given as follows.

(MP_u) The primal perturbed problem: find $x\in X$ such that

$\Phi(x,u)$=s-Inf $\cup_x \Phi(x,u)$

(MD_ω) The dual perturbed problem: find $\theta\in\Omega$ such that

s-$\Phi^c(\omega,\theta)(\omega(0)+\theta(0))$=s-Sup $\cup_\theta$ s-$\Phi^c(\omega,\theta)(\omega(0)+\theta(0))$

Definition 4.1

The point-to-set maps $P:U\to\underline{R}^p$ and $D:\Omega_X\to H$ defined by

P(u)=s-Inf $\cup_x \Phi(x,u)$ for $\forall u\in U$.

D(ω)(t)=s-Sup $\cup_\theta$ s-$\Phi^c(\omega,\theta)(t+\theta(0))$ for $\forall\omega\in\Omega$ and $t\in Y$

are called the primal and dual perturbed maps, respectively.

The following theorem shows how to describe (MP) and (MD) by means of P and D.

Theorem 4.4

(1)s-$P^c(\theta)$=s-$\Phi^c(0_X,\theta)$ for $\forall\theta\in\Omega$, that is, s-$P^c(\theta)(t)$=s-$\Phi^c(0_X,\theta)(t)$ for $\forall\theta\in\Omega$ and $y\in Y$.

(2) s-D*(x)=s-Φ^c*(x,0) for $\forall x\in X$.

(3) If $\Phi\in$s-$\Gamma^p{}_H(\Omega^*)$, in particular, as long as $\Phi(x,0)$=s-Φ^c*(x,0) for $\forall x\in X$, then s-D*(x)=$\Phi(x,0)$ for $\forall x\in X$.

ASSUMPTION (A1) For each $a\in\underline{R}^p$, there is $h_a\in H$ such that $h_a(0)=a$.

ASSUMPTION (A2) $\Phi\in$s-$\Gamma^p{}_H(\Omega^*)$ or Φ is s-$\Gamma^p{}_H(\Omega^*)$-normalizable at $(x,0)\in X\times U$ for each $x\in X$, i.e., s-Φ^c*(x,0)=$\Phi(x,0)$ for $\forall x\in X$.

Theorem 4.5

(1) s-Sup(MD)=s-P^c*(0), s-Inf(MP)=P(0).

(2) Under the assumptions (A1) and (A2), s-Inf(MP)=s-D*$^c(0_X)(0)$.

Now, we introduce two concepts: normality and stability.

Definition 4.2

(MP) is called s-(H,Ω)-stable if P is s-(H,Ω)-subdifferentiable at $u=0\in U$. Dually, (MD) is called s-(H,X)-stable if D is s-(H,X)-subdifferentiable at $\omega=0_X \in\Omega_X$.

Definition 4.3 (MP) is called s-(H,Ω)-normal if P satisfies P(0)=s-Pc*(0). Dually, (MD) is called s-(H,X)-normal if D satisfies D(0X)(0)=s-D*c(0X)(0).

Theorem 4.6

(1) (MP) is s-(H,Ω)-normal if and only if s-Inf(MP)=s-Sup(MD).

(2) Under assumptions (A1) and (A2), (MD) is s-(H,X)-normal if and only if s-Inf(MP) =s-Sup(MD) if and only if (MP) is s-(H,Ω)-normal.

Theorem 4.7

(1) If s-P^{c}*(0)=P(0), then s-Eff(MD)=s-∂_Ω P(0).

(2) Under assumptions (A1) and (A2), s-Eff(MP)=s-∂_X s-$D^{*c}(0_X)$. Moreover, if s-D^{*c}*$(0_X)(0)=D(0_X)(0)$, then s-Eff(MP)=s-∂_X D(0).

Theorem 4.8

(1) (MP) is s-(H,Ω)-stable if and only if s-Inf(MP)=s-Sup(MD)=s-Max(MD).

(2) Under assumptions (A1) and (A2), (MD) is s-(H,X)-stable if and only if

s-Inf(MP)=s-Sup(MD)=s-Min(MP)

Definition 4.4 The s-DGS(MP,MD)=s-Inf(MP)-s-Sup(MD) is called the s-(H,Ω) conjugate dual gap of (MP) and (MD), briefly called the dual gap.

From the results of the above several theorems, we have the following s-(H,Ω) strong duality theorem based on the s-(H,Ω)-stability or s-(H,Ω)-subgradient.

Theorem 4.9 (s-(H,Ω) Strong Duality Theorem)

(1) If (MP) is s-(H,Ω)-stable, then (MD) has the solution $\theta\in\Omega$, $\theta\in$s-Eff(MD). Moreover, (MP) is s-(H,Ω)-stable if and only if for there are x and θ such that $\Phi(x,0)$=s-$\Phi^c(0_X,\theta)(\theta(0))$. And if (MP) is s-(H,Ω)-stable and s-Min(MP)$\neq\varnothing$, then for each $x\in$s-Min(MP), there is $\theta\in\Omega$ such that $\Phi(x,0)$=s-$\Phi^c(0_X,\theta)(\theta(0))$

(2) (MD) has a solution, i.e., s-Eff(MD)$\neq\varnothing$, and s-DGS(MP,MD)=0 if and only if (MP) is s-(H,Ω)-stable. In this case, s-Eff(MD) =s-∂_Ω P(0).

Theorem 4.10 (s-(H,Ω) Inverse Duality Theorem)

Under assumptions (A1) and (A2), we have

(1) (MD) is s-(H,X)-stable if and only if s-Sup(MD)=$\Phi(x,0)$. In particular, if (MD) is s-(H,X)-stable, then s-Eff(MP)$\neq\varnothing$, and for each $\theta\in$s-Eff(MD), there is $x\in$s-Eff(MP) such that s-$\Phi^c(0_X,\theta)$ $(\theta(0))=\Phi(x,0)$.

(2) s-Eff(MP)$\neq\emptyset$ and s-DGS(MP,MD)=0 if and only if (MD) is s-(H,X)-stable. In this case, s-Eff(MP)=s-∂_X D(0_X).

Corollary Under assumptions (A1) and (A2), (MP) is s-(H,Ω)-stable and s-Eff(MP) $\neq\emptyset$ if and only if (MD) is s-(H,X)-stable and s-Eff(MD)$\neq\emptyset$. In this case, s-Inf(MP) =s-Sup(MD)=s-Max(MD)=s-Min(MP).

5. S-(H,Ω)-LAGRANGIAN MAPS

In this section, we shall introduce a s-(H,Ω)-Lagrangian map of the (MP) relative to the given perturbation Φ, and clarify the relationship between pairs of (MP) and (MD) and the weak saddle-points of the s-(H,Ω)-Lagrangian map.

Definition 5.1 The s-(H,Ω)-Lagrangian map L:X$\times\Omega\to$Rp of (MP) relative to Φ is defined by L(x,θ)=s-Φxc (θ)(θ(0)) for $\forall x\in X$ and $\theta\in\Omega$, where for each $x\in X$, Φx : $\Omega\to$Rp, Φx(u)=Φ(x,u) for $\forall u\in U$.

The (MP) and (MD) are represented by the s-(H,Ω)-Lagrangian map L as follows.

Theorem 5.1

(1) s-Φ^c(0_X ,θ)(θ(0))$\leq$=s-InfL_θ for $\forall\theta\in\Omega$, If H is closed s-Inf pointwise, then for $\forall\theta\in\Omega$, s-Φ^c(0_X ,θ)(θ(0))=s-Inf $\cup_x$ L(x,0) =s-InfL_θ .

(2) s-SupL_x $\leq$= s-InfΦ(x,0) for $\forall x\in X$. If Φ_x $\in$s-$\Gamma^p{}_H$ (Ω), then s-SupL_x =Φ(x,0). If Φ_x $\in$s-$\Gamma^p{}_H$ (Ω) for $\forall x\in X$, then for $\forall x\in X$, s-SupL_x =Φ(x,0).

Theorem 5.2 If H is closed s-Inf pointwise, and Φx $\in$s-ΓpH (Ω) for $\forall x\in X$, then the following conditions are equivalent to each other:

(1) (x,θ)$\in X\times\Omega$ is a strong saddle-point of L;

(2) x$\in$s-Eff(MP), $\theta\in$s-Eff(MD) and Φ(x,0)=s-Φ^c(0_X,θ)(θ(0));

(3)Φ(x,0)=s-Φ^c(0_X ,θ)(θ(0)).

Theorem 5.3 Under the Condition of theorem 5.2, if (MP) is s-(H,Ω)-stable, then the following conditions are equivalent to each other:

(1) x$\in$s-Eff(MP);

(2) there exists $\theta\in\Omega$ such that (x,θ) is a strong saddle-point of L. in this case, $\theta\in$s-Eff (MD).

Theorem 5.4 Under the condition of theorem 5.2, if assumptions (A1) and (A2) hold, and (MD) is s-(H,X)-stable, then the following conditions are equivalent to each other:

(1) $\theta\in$s-Eff(MD);

(2) there exists x$\in$X such that (x,θ) is a strong saddle-point of L. in this case, x$\in$s-Eff (MP).

6. CONCLUSIONS

In this paper, based on strong efficiency, we have extended the concepts of "conjugate", and "subdifferentiability" of functions to those of point-to-point maps, and developed a duality theory in multiobjective optimization.

One possible generalization of this paper is to consider the case that some cone "K" is introduced and the duality theory for multiobjective optimization with the domination structure with respect to the strong efficiency can be developed.

Finally, we hope to make contributions for solving multiobjective optimization problems in some sense by taking special cases for H and Ω.

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